The ecclesiastical calendar

Samuel Butcher

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ECCLESIASTICAL CALENDAR:

ITS THEORY AND CONSTRUCTION.

From / The Editors.

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ITS THEORY AND CONSTRUCTION.

BY

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PREFACE.

THIS work is the fruit of spare hours snatched from the last and busiest years of a laborious life. It was completed a few months before my dear Father's death, and the greater part of it had received his careful revision. The shape in which the book now appears is almost exactly that in which it was left by him: but what further and final changes he would have made, I cannot say. The duties of editing, which involved the verifying of many calculations, have fallen almost wholly on my brother, J. G. BUTCHER, as being more competent for the task than myself. But editorial care must at the best be a sorry substitute for the mind and eye of the Author.

S. H. BUTCHER.

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INTRODUCTION.

MANY valuable works have been written from time to time, and especially in recent years, on the Book of Common Prayer; and there are few questions relating either to the history of its compilation, the sources from which it was derived, or the *ationale* of its various parts, that have nót been investigated, and published in popular treatises, so as to be easily accessible to all who desire to make themselves acquainted with the subject. But there is one part of that Book which has been either wholly overlooked, or but slightly touched on, by its recent expositors (1), and respecting which the best sources of information are not readily accessible. I mean that part which contains the "Tables and Rules for the Moveable and Immoveable Feasts." The principles on which these Tables and Rules of our Ecclesiastical Calendar are framed, and even some of the technical terms employed in them, are, consequently, very imperfectly understood by many who are accurately enough acquainted with the various Offices themselves, and the Rubrics relating to their ministration. Wheatly, indeed, in his well-known work on the Book of Common Prayer, devotes a portion of his first chapter to the explanation of these Tables and Rules; but he does not discuss the subject with sufficient fulness or accuracy to make it clearly understood by a reader who has no further knowledge of it than what he can gather from those few pages. Now, as the subject is one about which the Clergy, at least, as the professional expounders of our Prayer Book, may be expected to possess some amount of information, I have endeavoured, in the following pages, to supply it (2). It is not my purpose to write a complete treatise on the Ecclesiastical Calendar. This would require a very large volume, and, moreover, would involve discussions which, however interesting in themselves, are not necessary for the main object which I have in view: which is, to explain to the careful student of the Prayer Book the principles on which this part of it is based, and to point out in a general way the application of these principles to the construction of the Tables and Rules which it contains, and to the solution of some of the various questions connected with the Ecclesiastical reckoning of time. A fair acquaintance with Arithmetic on the part of the reader is, of course, indispensable. Some knowledge of the rudiments of Astronomy I have also presupposed; and for the sake of those who are familiar with the simpler algebraical operations, I have occasionally given some of the more important formulæ in which the tabular details are concisely summed up (3).

(1). In Campion and Beamont's very useful work, "The Prayer Book Interleaved," there are some valuable remarks on the Ecclesiastical Calendar, borrowed from a paper on the subject, by Professor De Morgan, in the "Companion to the British Almanac" for the year 1845. They are, however, suited only for those who have already some acquaintance with the subject; and moreover, they relate exclusively to that part of it which is specially connected with the determination of Easter.

(2). It has, indeed, been proposed by the English "Ritual Commission," and by the General Synod of the Church of Ireland, to omit the three "General Tables for finding the Dominical Letter, and the places of the Golden Numbers in the Calendar," as being unsuited, from their intricacy, to the ordinary reader. Still, as these Tables are ratified by an English Act of Parliament (24 Geo. II., ch. 23), and also by an Irish Act (21 and 22 Geo. III., ch. 48), some acquaintance with their nature and structure may fairly be required from the Clergy of both Churches. It may be added that even the remaining Tables and Rules, which it is proposed to retain in the Prayer Book, demand a fuller and more detailed explanation than can easily be obtained by an ordinary student.

(3). Some useful popular information on the subject of the Ecclesiastical Calendar will be found in *Wheatly's* work, above referred to; *Sir H. Nicolas'* Chronol. of History; *Smith's* Dict. of Antiquities, Art. "Calendar"; *Dr. Stephens'* edition of the Book of Common Prayer, Vol. I. pp. 264, *sq.*; *Sir J. Herschel's* Astronomy (11th edit. 1871); and *Dr. Lardner's* Museum of Science and Art, Vols. V. and VII. Those who desire to study the subject more deeply will do well to consult the Art. "Calendar" in the Encyclop. Britann., 7th edit.; the great French work, "L'Art de verifier les Dates"; *Delambre*, Astron. Mod., Tome I.; also his Astron. Theor. et Prat., Tome III.; *Gassendi*, Roman. Calend. compend. exposit., Oper. Tom. V.; *Ideler*, Handbuch der Math. u. Techn. Chronologie; *De Morgan*, Companion to the British Almanac, 1845; with his Book of Almanacks, 1851; *Francœur*, Theorie du Calendrier; *Herzog*, Real-Encyclop., Arts. "Kalender," and "Christ. Zeitrechnung." Above all, the great work of *Clavius*, Roman. Calendar. a Gregor. XIII. restitut. explicat., Romæ, 1603, is indispensable in order to a complete understanding of the whole matter.

THE

CIVIL AND ECCLESIASTICAL CALENDAR.

1. The Festivals and Holydays which the Church observes fall into two general classes, viz.: the Moveable, and the Immoveable. The latter, as their name expresses, have fixed places in the Calendar; that is to say, they occur on certain fixed days of the month. For example, the *Circumcision* always falls on the 1st of January; the *Epiphany*, on the 6th of January; the *Annunciation*, on the 25th of March; *St. Andrew's Day*, on the 30th of November; *Christmas Day*, on the 25th of December.

The Moveable Feasts, on the other hand, are those whose places in the Calendar are not fixed, but variable within certain limits. They fall on fixed days of the week, but not on fixed days of the month. *Sunday* is a Moveable Festival. Consequently, so is Easter Day, with the whole group of Festivals depending upon it ([1]), which are enumerated in the Prayer Book. The Moveable and Immoveable Feasts may, therefore, be called the Planets and Stars of the Ecclesiastical Calendar ([2]).

(1). "Duo Festorum sunt genera, quorum unum stabilem in Calendario locum retinet, complectiturque dies qui semper statis anni temporibus recurrunt, ideoque *immobiles* dici solent. Alterum sedem habet mobilem, certis tamen dierum terminis conclusum, comprehenditque sacra diei Festi Paschalis, atque ex eo pendentium Festorum dierum, solemnia, qui idcirco *immobilia* appellantur. Ad Festa immobilia celebranda usus tantum mensium solarium necessarius est; ad Festa mobilia, et menses solares et lunares necessarii sunt."—*Clavius*, Rom. Cal. Expl., Procem., p. 1.

(2). Speaking generally, the same day of the week does not fall on the same day of the month in different years. But after the lapse of a certain period, or cycle, the coincidence recurs, and continues through the corresponding years of each cycle.—*Vid.* Art. 26, on the Solar Cycle.

2. The Moveable Feasts, again, fall into two groups: (*a*) the Sundays and other Festivals immediately depending on Easter Day; (*b*) all the remaining Sundays of the year. The former group embraces nine Sundays before, and eight after, Easter

Day. Of these the nearest to Easter, before and after, are Palm Sunday, and Low Sunday (1); the farthest from it, before and after, Septuagesima, and Trinity Sunday, respectively. Besides these seventeen *Sundays*, the following Holydays, which have Collects, Epistles, and Gospels, also depend on Easter: viz., Ash Wednesday, and the six days of Passion Week; Monday and Tuesday in Easter Week; Ascension Day; Monday and Tuesday in Whitsun Week. So that the total number of Sundays and other Holydays depending on Easter (including Easter Day itself) amounts to *thirty*.

(1). The terms *Palm* Sunday and *Low* Sunday (first after Easter) do not occur in the Prayer Book. But the first Sunday in Lent is called also *Quadragesima;* and the fifth after Easter, *Rogation* Sunday.

3. As the total number of Sundays in the year is fifty-two or fifty-three (1), it follows from what has just been said, that the number of Sundays not *directly* dependent on Easter is thirty-four or thirty-five (2). The Sunday before the Epiphany, when there is one, is called the first, or second, Sunday after *Christmas*. The Sundays between the Epiphany and Septuagesima are called Sundays after the *Epiphany*. Those between Trinity and Advent are called Sundays after *Trinity* (3). The four Sundays preceding Christmas Day are named from *Advent* (4). And, lastly, the Sunday, or Sundays, between Christmas Day and the Epiphany are called the first, or second, Sunday after *Christmas Day*.

(1). In a common year there will be 53 Sundays, if the year begins on Sunday; and in a leap year, if the year begins on Saturday or Sunday: because a common year contains 52 whole weeks + 1 day $\left(\frac{365}{7} = 52 + 1\right)$; in other words, it ends on the same day of the week on which it begins. In a leap year we have $\frac{366}{7} = 52 + 2$; therefore 53 Sundays, if the 1st or 2nd of January is Sunday.

(2). Strictly speaking, as the number of Sundays after the Epiphany, and after Trinity, depends on the position of Easter Day, these Sundays also may, in this sense, be regarded as dependent on Easter. The only Sundays absolutely independent of Easter are those connected with Advent and Christmas; in other words, those which occur from Nov. 27th to Jan. 6th, both inclusive; the maximum number of which Sundays is six, the minimum five:—six, when Advent Sunday falls on any day before Dec. 3rd; five, if on Dec. 3rd. It must be remembered that, by the Prayer Book definition of *Advent Sunday*, it may fall on any day of the month between Nov. 27th and Dec. 3rd. If Advent Sunday fall on Nov. 27th, Christmas Day will be on Sunday; and so also will the following New Year's Day. If Advent Sunday fall on Dec. 2nd, the following Epiphany will be on Sunday. In order that there may be two Sundays after Christmas, Advent Sunday must fall on one of the days, Nov. 28th to Dec. 1st; in other words, if it fall on Nov. 27th, or Dec. 2nd or 3rd, there will be but *one* such Sunday.

(3). In the Greek and Roman Churches, Trinity Sunday is not so named, but is called the first Sunday after *Pentecost;* and the following Sundays, to Advent, are also reckoned as after Pentecost. The reason of this is, that the institution of a special Feast in honour of the Holy Trinity is comparatively recent, having been first enjoined by the Synod of Arles, A. D. 1260. In ancient times all Sundays were regarded as commemorating this mystery. The name "Trinity Sunday" has been in use in the English Church since the eleventh century, when Osmund, Bishop of Salisbury, remodelled the Offices (A. D. 1085). Our Reformers followed the Sarum Missal in adopting the term "Trinity Sunday," and numbering the following Sundays from it.

(4). Our present "twenty-fifth Sunday after Trinity" ought also to be named in reference to *Advent*, and termed "The Sunday next before Advent;" and so the General Synod of the Church of Ireland has agreed to call it.

4. The *least* number of Sundays "after the Epiphany" is *one;* the *greatest* number, *six*. The minimum occurs when Easter falls, in a *common* year, on either of the three days, 22nd, 23rd, or 24th of March (the 22nd being the *earliest* day on which it can fall) (1); in a *leap* year, on either the 22nd or 23rd of March (2). That there may be *six* Sundays after the Epiphany, Easter Day must fall, in a *common* year, on either of the four days, 22nd, 23rd, 24th, or 25th of April (the 25th being the *latest* day on which it can fall); and in a *leap* year, on any of the five days, 21st–25th of April (3). The *minimum* occurs very rarely. In the present century it has happened only three times: viz., in 1818, 1845, 1856; and it will not occur again until 1913. The *maximum* has taken place five times this century: viz., in 1810, 1821, 1832, 1848, 1859; and will again occur twice, viz., in 1886 and 1889. Again, the number of Sundays after Trinity is *twenty-two;* the *maximum*, *twenty-seven*. The *maximum* occurs when Easter falls on any of the five days from 22nd to the 26th of March: the *minimum*, when it falls on the 24th or 25th of April (4). The *minimum*, like the minimum number of Sundays after the Epiphany, happens very rarely. In the present century it has occurred but once (1859); and it will occur but once again (1886). The *maximum* happens more frequently. It has taken place already five times this century (1815, 1818, 1837, 1845, 1856), and will again take place twice (1883, 1894) before its close.

The sum of the Sundays "after the Epiphany" and "after Trinity" cannot be less than twenty-eight, nor greater than twenty-nine. The *maximum*(29) occurs when Easter Day falls on any of the ten days, March 25, 26; April 1, 2, 8, 9, 15, 16, 22, 23; and the Sunday Letter in each of these cases will be G or A. All this is obvious on inspection of the Prayer Book "Table to find Easter Day," and the "Table of Moveable Feasts according to the several days that Easter can possibly fall upon" (5).

(1). This necessarily follows from the definition of *Easter Day* given in the Prayer Book. A full explanation of the matter will be given in the proper place (Art. 59.)

(2). From the Epiphany to the 22nd of March (the earliest day on which Easter can fall) there are (both exclusive), in a common year, 74 days, and in a leap year 75; that is to say, 10 weeks, *plus* 4, or 5, days. But from Easter to Septuagesima, inclusive, there are 9 weeks—leaving 1 week and 4, or 5, days between Septuagesima and Epiphany. Therefore, in this extreme case, there can be but *one* Sunday between them. It also appears from this that, in a common year, Easter might fall *two* days later ($4 + 2 = 6$) than the 22nd of March; or, in a leap year, *one* day($5 + 1 = 6$), without involving a second Sunday between Epiphany and Septuagesima.

(3). From the Epiphany to April 25th (the latest day on which Easter can fall), there are (both exclusive), in a common year, 108 days, and in a leap year, 109; that is, 15 weeks, *plus* 3, or 4, days. Deducting 9 weeks, as before, we get *six* weeks, *plus* 3, or 4, days between Septuagesima and Epiphany. And these *six* weeks will still intervene, even though Easter should, in a common year, fall 3 days, and, in a leap year, 4 days, before the 25th of April.

(4). The earliest day on which Trinity Sunday can fall, corresponding to the earliest Easter Day (March 22), is May 17, whose *Calendar* Letter (see Art. 23) is D: accordingly, when Trinity Sunday falls on this day, Advent Sunday must fall on Nov. 29 (Letter D). The number of *intermediate* days between May 17 and Nov. 29 is 195, = 27 weeks and 6 days. Hence the maximum number of "Sundays after Trinity" is 27. And as Advent Sunday *may* fall on any of the four following days (Dec. 3 is the latest), it follows that there will still be 27 Sundays, even though Easter should fall on any of the four days after the 22nd of March. There may, then, be 27 such Sundays, but no more: and the Sunday Letters must be D, E, F, G, A. Again, the latest day on which Trinity Sunday can fall, corresponding to the latest Easter Day (April 25), is June 20, the Calendar Letter of which is C. Therefore, when Trinity Sunday falls on this day, Advent Sunday must fall on Nov. 28 (Letter C). The number of intermediate days between June 20 and Nov. 28 is 160, = 22 weeks, and 6 days. Consequently, the *minimum* number of Sundays "after Trinity" is 22. And, as Advent Sunday cannot fall earlier than Nov. 27, it follows that there would still be only 22 weeks if Easter fell on April 24, but not earlier. Easter Day must, then, in this case, fall on April 24 or 25; and the Sunday Letter must, accordingly, be B or C.

(5). That the number of Sundays must be 28 at least, or 29 at most, may be easily shown directly. From the Epiphany (Jan. 6) to the earliest Advent Sunday (Nov. 27), there are (both exclusive) 324 (leap year, 325) days; that is, 46 weeks, *plus* 2, or 3, days. Again, from the Epiphany to the latest Advent Sunday (Dec. 3), there are 330 or 331 days; = 47 weeks, *plus* 1, or 2, days. Therefore, subtracting the 18 Sundays from *Septuagesima* to Trinity Sunday (both inclusive), we get, for the *minimum* remainder, 28; and for the *maximum*, 29.

Proceeding in a similar way with the rest of the *thirty-five* days on which Easter can possibly fall, it is easy to see that, in *common* years, the number of Sundays after Epiphany fall into the following groups:—

When Easter falls	on any	of the	*Three*	days,	March	22–24,	there will be	*One*	such	Sunday.
,,	,,	,,	*Seven*	,,	,,	25–31,	,,	*Two*		,,
,,	,,	,,	,,	,,	April	1– 7,	,,	*Three*		,,
,,	,,	,,	,,	,,	,,	8–14,	,,	*Four*		,,
,,	,,	,,	,,	,,	,,	15–21,	,,	*Five*		,,
,,	,,	,,	*Four*	,,	,,	22–25,	,,	*Six*		,,

All this is obvious on inspection of the Prayer Book "Table of the Moveable Feasts, according to the several days that Easter can possibly fall upon." This Table is calculated for *common* years. Therefore, the Note appended to it respecting Leap Year must be carefully attended to:—"In a Bissextile or Leap Year, the number of *Sundays* after Epiphany will be the same as if *Easter Day* had fallen one day later than it really does. And for the same reason, one day must, in every leap year, be added to the day of the month given by the Table for *Septuagesima* Sunday. The like must be done for *Ash Wednesday*, unless the Table gives some day in March for it; for, in that case, the day given by the Table is the right day." The part of the Rule relating to the Epiphany and Septuagesima is obvious; because, supposing, for example, that in any leap year, Easter Day falls on March 30, the insertion of the intercalary day that year makes the number of days (and therefore, of Sundays) from Epiphany to Easter the same as it would be in a common year in which Easter Day fell on March 31. But the number of Sundays from Septuagesima to Easter is invariable, viz., 9. Therefore, the number of Sundays between Epiphany and Septuagesima, in leap year, is the same as if Easter fell a day later in a common year. Again, as the number of days between Easter and Septuagesima is invariable (63), and as Septuagesima always falls before Feb. 29, the insertion of the intercalary day in leap year necessarily brings down Septuagesima one day lower in the month. The same will be true of Ash Wednesday also, unless it fall in March; for, in that case, the intercalation will have taken place before it, and it will be circumstanced, as regards Easter Day, as if the year were a common one.

5. The *Immoveable* Feasts appointed to be observed in our Prayer Book are twenty-four in number.* Of these, *Three* commemorate events in our Lord's personal history; viz., the Nativity, the Circumcision, and the Epiphany. *Two* commemorate events in the life of the Virgin Mary, as connected with our Lord's history, viz., the Annunciation and the Purification. *Ten* are assigned to the commemoration of the twelve Apostles, four of whom are united in pairs: scil., St. Philip and St. James; St. Simon and St. Jude. *Two* commemorate the non-apostolic Evangelists, St. Mark and St. Luke. The rest commemorate, respectively, the miraculous birth of John the Baptist (the only nativity commemorated, except that of our Lord); the death of the Proto-martyr, St. Stephen; the Conversion of St. Paul, unique of its kind; St. Barnabas, the companion of the great Apostle of the Gentiles; the murder of the Innocents, in connexion with our Lord's Nativity; St. Michael and all Angels; and, lastly, All Saints. The last two Festivals relate to the Church in Heaven, as the others to the early history of the Church on Earth.

The Collects, Epistles, and Gospels for the Immoveable Feasts are placed in the Prayer Book after those for Sundays; except those belonging to the six composing the Christmas group of Feasts, which are all placed together after the fourth Sunday in Advent; viz., Christmas Day, St. Stephen, St. John the Evangelist, the Holy Innocents, the Circumcision, and the Epiphany.

The order in which the first seven of the Saints' Days are arranged in the Calendar (it being remembered that the church year commences with the Advent season) is worthy of remark. St. Andrew (Nov. 30) comes first, as being the Apostle who was first called by Christ. St. Thomas (Dec. 21) is the next, on account of his noble con-

* I have not included the Day of the Queen's Accession; which is the only State Holyday now appointed in the Prayer Book.

fession of his risen Master. Immediately after the Nativity, the foremost place is assigned to St. Stephen as the Proto-martyr; the next to St. John, as the beloved Disciple. Then follows the Innocents' Day, because of their connexion with the birth of Christ. After that comes the conversion of St. Paul, the great Apostle of the Gentiles; and, lastly, St. Matthias' Day, he being the Disciple elected to fill up the vacancy in the Apostolic College caused by the apostasy of Judas.

6. The number of Sundays in the year being 52 or 53; and the other days depending on Easter, for which Collects, Epistles, and Gospels are provided in our Prayer Book, being 12; it follows that the total number of *Moveable* Holydays observed by our Church is 64 or 65. Adding to these the 24 *Immoveable* Festivals, we see that the total number of days for which public service is appointed (not including the Queen's Accession) amounts to 88 or 89. In other words, the Church appoints Divine Service for nearly one-fourth of the entire year, one-seventh being Sundays, and more than a ninth other days.

7. The two following Tables will be found useful in making the foregoing and similar calculations.

TABLE I.

To find the Number of Days from Easter (exclusive) to some of the principal Festivals depending upon it (inclusive).

To . . .	Septuag.	Ash Wed.	Rogat. Sun.	Ascen. Day.	Whit-Sun.	Trin. Sun.
From Easter Day,	-63^d	-46^d	$+35^d$	$+39^d$	$+49^d$	$+56^d$

The number of days from any of the dependent Festivals (exclusive) to any other (inclusive) will be the sum or difference of the corresponding numbers, according as they have opposite or the same signs, *e. g.*, from Septuagesima to Trinity Sunday there are $63 + 56 = 119$ days; from Septuagesima to Ash Wednesday, 17 days; and from Ascension Day to Trinity Sunday, 17 days.

It may be further noticed that the Thursday after Trinity Sunday (the Romish Festival of *Corpus Christi*) being 60 days after Easter, is 61 days after Easter-eve. Hence it follows that from Easter-eve to Corpus Christi there are always exactly two calendar months.(1)

TABLE II.

TO FIND THE NUMBER OF DAYS ELAPSED FROM ANY DAY OF THE YEAR TO ANY OTHER DAY, ONE BEING EXCLUSIVE, THE OTHER INCLUSIVE.

Number of Days *elapsed* from Jan. 1st to the first of each Month.

Date.	In a Common Year.	In a Leap Year.	Date.	In a Common Year.	In a Leap Year.
Jan. 1, . . .	0	0	July 1, . . .	181	182
Feb. 1, . . .	31	31	Aug. 1, . . .	212	213
March 1, . .	59	60	Sept. 1, . . .	243	244
April 1, . .	90	91	Oct. 1, . . .	273	274
May 1, . . .	120	121	Nov. 1, . . .	304	305
June 1, . . .	151	152	Dec. 1, . . .	334	335

This Table will also show the interval (as above defined) between *any* two days of the year.

Ex. 1. From Jan. 6 to March 22.

Jan. 6 to March 6 (= Jan. 1 to March 1) = 59; Leap year, 60 days.
plus . . . . 16 . . . 16 „

75 or 76 „

Ex. 2. From March 22 to Nov. 1.

March 22 to Nov. 22 (= March 1 to Nov. 1) = 245 days.
minus . . . . 21 „

224 „

It is only when the interval involves a *bissextile* February that the number of days is increased by 1.

Ex. 3. From Feb. 5 to Dec. 1, 1872 (Leap year).

Feb. 5 to Dec. 5 (= Feb. 1 to Dec. 1) = 304 days.
minus . . . . 4 „

300 „

(1). Let π denote the date of Easter Day. Then Easter Eve $= \pi - 1$; and *Corpus Christi* $= \pi - 1 + (30 + 31)$. But if Easter fall in *March*, the following two months, April and May, will contain 30 + 31 days; and if Easter fall in April, the two next months, May and June, contain 31 + 30 days; so that in both cases the interval between Easter Eve and *Corpus Christi* is exactly two calendar months.

8. To determine the days on which the Moveable Feasts fall in any year is the object of "The Tables and Rules" given in the Prayer Book. The first step obviously is to ascertain on what days of the year the *Sundays* fall; or, rather, any one Sunday: because, one being known, all the rest follow from it, as a matter of course. This is effected by means of the *Dominical,* or Sunday, *Letter*, the mode of finding which shall presently be explained. The Sundays in any year being determined, *two* of them, viz., Easter Day and Advent Sunday, are further marked out by the special definitions given of them in the Prayer Book; *scil.* :—

"*Easter Day is always the first Sunday after the Full Moon which happens upon, or next after, the twenty-first of March; and if the Full Moon happens on a Sunday, Easter Day is the Sunday after*" (1).

"*Advent Sunday is always the nearest Sunday to the Feast of St. Andrew* (Nov. 30), *whether before or after*" (2).

(1). The ambiguity of this definition, and its more accurate form, will be pointed out hereafter. (Art. 45.)

(2). It should be added, "or the day of that Feast itself." Advent Sunday falls on the 30th of Nov. as often as on any other of the 7 days over which it extends—viz., five times every 28 years. The intervals at which each of these 7 days recurs, during the Cycle of 28 years, are two of 6 years each, one of 5, and one of 11. The interval of 12 years occurs very rarely. In the case of Nov. 30 (Sun. Lett. E), it will happen but three times during the eight centuries from 1800 to 2500; namely, in 1902, 2302, 2510. The *unequal* subdivision of the Cycle of 28 years arises from the interposition of the leap years, as will appear more clearly when we come to speak of the Solar Cycle (Art. 26).

9. *To find on what days of any given year the Sundays fall*, it will be useful to premise a few observations respecting the *Solar Day*, and the *Solar*, or *Tropical*, *Year*. The determination of *Easter* Sunday is a more difficult problem, as it involves the consideration of the *Lunar*, as well as the *Solar*, year, and the relation between them. (*Vid.* Art. 40.)

From the Sun's apparent motion we derive our *two standard units of Civil Time*, namely, *the Solar Day, and the Tropical Year.* The Solar day is the interval between two consecutive transits of the centre of the Sun's disc across the meridian; or, to speak more correctly, the interval between two consecutive returns of the same terrestrial meridian to the centre of the Sun's disc. But as it is an essential condition of a standard unit that it should be invariable; and as the actual Solar day, as just defined, is not so (1), it becomes necessary to employ, in place of the *real* Solar day, what is called the *mean Solar day;* that is to say, the average or mean of all the real Solar days throughout the year (2). This mean Solar day is divided into twenty-four hours, each hour being subdivided into minutes and seconds (3). Clocks, and other artificial time-keepers, are constructed so as to keep, and show, mean time.

(1). Astronomy shows that the want of uniformity in the length of the true Solar day arises from two causes, namely, the obliquity of the Ecliptic, and the ellipticity of the Earth's orbit.

(2). In order to represent more clearly the relation between the (apparent) actual and mean motion of the Sun, it is usual to imagine the existence of a fictitious sun, whose daily (easterly) angular motion in the heavens is perfectly uniform, and equal to the average daily angular motion of the real Sun, namely, 0° 59′ 8·2″. The place of this imaginary sun never differs much from that of the real Sun. The time measured by its motion is called *mean time;* and the moment when its centre crosses the meridian is called *mean noon.* On the other hand, the time indicated by the real Sun is called *apparent time:* and the moment of its transit across the meridian is *apparent noon.* A well-regulated clock, as above said, shows mean time; a correctly mounted sun-dial shows apparent time. The difference between them is called the *Equation of time.*

(3). This use of the word 'day,' in which it denotes 24 hours, differs, it is needless to observe, from the popular use of the word, in which it is opposed to 'night,' and expresses the interval during which we daily receive light from the Sun. The Greeks employed the word *νυχθήμερον* to denote the whole duration of 24 hours, including a day and night. The English language has no corresponding term. The *Hebrew* day was literally a *νυχθήμερον*, as it began at sunset, so that the night *preceded* the day. This arrangement is at least as old as the time of Moses, being expressly alluded to in the first chapter of Genesis. Most modern nations, like the ancient Egyptians, make the civil day to commence at midnight, reckoning 12 morning hours from midnight to midday, or noon; and 12 evening hours, from noon to midnight. Astronomers, on the other hand, following the example of the great Alexandrian astronomer, Ptolemy, commence their day at noon, and reckon on through the whole 24 hours.

10. We have just seen that, as the natural and true Solar day is of variable length, it has been found necessary to substitute in place of it, for the ordinary purposes of civil life, the mean Solar day. There is, indeed, another natural interval of time which does possess, in the highest degree, the property of invariability, but which, nevertheless, is not adapted to those purposes. I mean what is called the *Sidereal Day;* that is to say, the interval between two successive transits of the same fixed *star* across the meridian; in other words, the time of one complete revolution of the Earth on her axis. This interval is not suited to be an ordinary measure of time, because it is neither conspicuous enough, nor capable of easy and universal observation (1).

(1). The Sidereal day is, so far as actual observation extends, absolutely invariable. Laplace calculated that since the time of Hipparchus, an interval of twenty centuries, it has not altered the 100th part of a second. But the more recent opinion is, that the time of the Earth's rotation is gradually, though imperceptibly, diminishing, by reason of the friction connected with the tidal motions. Assuming (as has been supposed) that the *accumulated* amount of this retardation is 12 seconds in a century, the result is that the day is about the 64th part of a second shorter now than it was 2500 years ago. The question is, in fact, to find the common difference (d) of an arithmetical series, consisting of n terms, whose sum (S) is given, and whose first term = com. diff.

In this case, $S = \left(2d + \overline{n-1}\,d\right)\frac{n}{2} = \frac{d(n+1)n}{2}$.

In the question before us, $S = 12$ seconds; d = the daily retardation; n = number of days in 100 years = 36,524 days.

Hence we get $24^s = \{36{,}524 + (36{,}524)^2\} \times d$;

$$\therefore\ d = \frac{24^s}{1334002576} = \cdot 0000000171^s = \text{daily retardation.}$$

Multiply this by 913100 (the number of days in 25 centuries) and we get $\cdot 01561401^s = \frac{1}{64}$th part of a second, nearly.

From what was said above respecting the mean Solar day it is evident that the Sidereal day is *shorter* than it. The difference is the length of time which the mean sun takes to traverse an arc of 0° 59′ 8·2″, that is to say, 3 min. 55·91 sec. Consequently, the length of the Sidereal day, in mean Solar time, is $23^h\ 56^m\ 4{\cdot}09^s$; the Sidereal hours, minutes, and seconds being in the same ratio. Reduced to decimals, the ratio of the mean Solar and Sidereal days is as 1·002738 to 1; in other words, a Sidereal day = Solar day × ·9973. The same thing may be thus expressed: the time of the Earth's rotation falls short of 24 common hours by $3^m\ 55{\cdot}91^s$.

11. The second natural unit of civil time, above alluded to, is called the *Solar*, or *Tropical*, *Year*. It is the interval between two consecutive passages of the Sun's centre through the same equinoctial point. As the intersection (or node) of the Equator and the Ecliptic has a slow retrograde motion—that is, a motion in a direction *opposite* to the apparent annual course of the Sun—it is obvious that the Sun, after passing through either equinoctial point (or node), will meet it again *before* the completion of his annual revolution in the Ecliptic. The mean annual amount of this retrograde motion (*the Precession of the Equinoxes*, as it is called) is 50·1″—a very minute quantity in itself, but which, by its accumulation from year to year, becomes at length very considerable. In the course of 2200 years, the Equinox has retrograded more than 30° (1). The corresponding *mean length of the Tropical Year* is, according to the "British Nautical Almanac," $365^d\ 5^h\ 48^m\ 47{\cdot}46^s$, or, reduced to decimals, $365^d{\cdot}242216$ (2). This mean length itself is subject to a very small periodic oscillation. It is now about $4{\cdot}21^s$ shorter than in the time of Hipparchus (3).

(1). The period in which it performs a complete tour of the Ecliptic is 25,868 years. The physical cause of Precession is the rotation of the Earth combined with its spheroidal form, and the consequent unequal attraction of the Sun and Moon on its equatorial and polar regions.

(2). *Vid.* Sir John Herschel's Astron., ed. xi., p. 690. Other astronomers vary a little from this in the seconds. Delambre gives $51{\cdot}6^s$; Ideler, after Lalande, 48^s. Bessel agrees very nearly with the English; he gives the seconds 47·59; so that the years = 365·2422175.

(3). The cause of this small variation is the periodical secular diminution in the obliquity of the Ecliptic, resulting from the joint attraction of all the planets. The amount of it is about 48 seconds in a century. The diminution will go on for many ages, after which it will cease, and the obliquity will again increase. Thus, the plane of the Ecliptic will oscillate backward and forward, the extent of deviation on either side of the mean position being less than 1° 21′.

12. Although the length of the Tropical year is not invariable, still, from its connexion with the Seasons (1), it has been universally adopted as the larger unit of civil time, rather than the *Sidereal year*, which, like the Sidereal day, does, indeed, possess the property of invariability, but wants the essential condition of adaptability to the ordinary purposes of life. The Sidereal year is the interval between two successive conjunctions of the Sun with the same fixed star in the Ecliptic; in other words, the time of the Sun's performing a complete revolution in his apparent orbit. We have already seen that the annual regression of the equinoctial points is 50·1″; and this arc is described by the Sun in $20^m\ 19{\cdot}9^s$. Consequently, the time taken by him to arrive again at the same fixed star from which he is supposed to start will be so much longer than the Tropical year. Hence, the *length of the Sidereal year, reckoned in mean Solar time*, is $365^d\ 6^h\ 9^m\ 9{\cdot}6^s$; or, expressed in decimals, $365^d{\cdot}2563612$ (2).

(1). The Equinoxes and Solstices are very remarkable, and (except at the Equator) universally observable, phenomena. They were, therefore, very naturally taken as defining four corresponding points in the annual course of the Seasons; with which, especially in temperate latitudes, these celestial phenomena had very obvious relations. The Sidereal year, on the other hand, though, on close observation, susceptible of a similar application to the purposes of life, yet does not present such well-marked and striking points of distinction as the Tropical.

(2). The length of the Sidereal year, reckoned in *Sidereal* time, is one whole day more—namely, $366^d\ 6^h\ 9^m\ 9{\cdot}6^s$. The reason of this difference is that the apparent annual motion of the Sun among the stars is in a contrary direction to the apparent diurnal motion of both Sun and stars, the result of which is that the Sun seems continually to drop behind the stars in his daily course. In the course of a year, he will have fallen behind them by a whole circumference of the heavens; that is to say, he will have made in a year one diurnal revolution less than the stars: so that the same interval of time which is measured by $366^d\ 6^h$, &c., of Sidereal time, will be called $365^d\ 6^h$, &c., if reckoned in mean Solar time. This, again, gives us the ratio already found between the mean Solar (or civil) and the Sidereal day: viz., 1·00273791 : 1. In other words, 100,000,000 common, or civil, days are equal to 100273791 Sidereal days; or, less exactly, in 1000 common days, the Earth makes 1002¾ rotations on her axis.

13. The Solar day and Tropical year, however convenient in other respects as standard units of time, labour under this disadvantage, that they are incommensurable quantities: that is to say, the Tropical year does not consist of an exact number of Solar days, nor is the overplus fraction an exact sub-multiple of a day, such as a fourth, for example. Now, as the year must, for the practical purposes of life, contain an *integer* number of days, the plan that has been generally adopted is to take the next less integer as the normal length of the year, and to let the fractional remainder accumulate each

year until the accumulation exceeds a whole day, and then add this day to the length of the corresponding year. The day so added is called the *intercalary day*, and the year to which it is added the *intercalary year*. This method amounts, in effect, to employing *two* civil years of different lengths: the normal one (of 365 days), which is less than the Tropical year; the other (of 366 days), which is greater than it. The main object of a well-adjusted Calendar is so to arrange the succession of these two years of 365 and 366 days, respectively, that the connexion between the Solar day and the Tropical year may be continually maintained, in a manner at once the most exact, simple, and convenient [1].

(1). Among the most remarkable and famous adjustments of the Solar year, that adopted by the ancient Egyptians holds a conspicuous place. Their year consisted of 365 days, divided into 12 months, of 30 days each; the remaining five days being annexed (ἐπαγόμεναι) at the end. The French Revolution Calendar was constructed on the same principle, except that the intercalary day, which the Egyptians did not admit at all, continued to be inserted every fourth year. The Egyptians were, no doubt, fully aware that the Tropical year contained about six hours more than 365 days; but this number being, probably, consecrated by early religious associations, they were unwilling to correct it. The result, of course, was that the commencement of the Egyptian year was not fixed in reference to the Seasons, but retrograded at the rate of nearly six hours annually, and so traversed the whole circle of the Seasons in 1460 years. In other words, at the end of 4×365 years, the annual error of $\frac{1}{4}$th of a day amounted to 365 days, or a whole year; and the commencement of the year was restored to the same place that it occupied at the beginning of the period of 1460 years. From this perpetual change, the Egyptian year was called *annus vagus*. The Arabians, on the other hand, having learned from the Egyptians the true (approximate) length of the Tropical year, adopted the intercalation of a day every fourth year; so that 1460 Arabian years were equivalent to 1461 Egyptian. This gave rise to a difference between the dates of these two ancient nations, similar to the difference which at present exists between the Russians and other European nations. The restoration in Egypt of the coincidence between the Civil and Solar year at the end of the 1460 years gave rise, as is known, to the fable of the Phœnix.—*Vid.* Ideler, "Handbuch," I. 68, 93.

14. Of all the methods of maintaining, by means of intercalation, the correspondence between the Civil and Solar years, that established by Julius Cæsar in his reformed Roman Calendar is the most worthy of note; not merely because this Calendar became that of the Roman Empire, but, still more, because the Julian Calendar was adopted, with some slight changes and adaptations to Church purposes, by the early Christians in the West, and still continues, with the corrections introduced into it by Pope Gregory XIII., in the year 1582, to be the Civil and Ecclesiastical Calendar of Western Christendom.

15. The ancient Roman year, as arranged by Numa, who probably learned it from the Greeks of Magna Græcia, was *Lunar* [1]. And Livy informs us, in a remarkable passage [2], that Numa so regulated his Lunar year of twelve months, by the insertion of intercalary months, that at the end of every *nineteenth* [3] year it again

coincided with the same point of the Sun's course from which it started. The mention of this nineteen-year cycle (if *vigesimo* be the correct reading) seems to throw doubt on the correctness of Livy in attributing the use of it to Numa, as we know it was first introduced at Athens by Meton, B. C. 432. And, in fact, some learned men maintain that the so-called Calendar of Numa was not his at all, but the work of the *Decemvirs*, who were sent to Athens (U. C. 300) to transcribe the Laws of Solon, and to acquaint themselves with the constitutions and usages of the other states of Greece. The Athenians at that time made use of the *Octaeteris*, or eight-year cycle, for adjusting the Solar and Lunar year; and the Decemvirs, most probably, became acquainted with it, and on their return to Rome employed it in reforming the previously-existing Calendar, by changing the year from a purely Lunar one to a Luni-Solar (Ideler, II. 56, *sq.*). The introduction of the intercalary month, Mercedonius, was probably occasioned by this change (4). However, the Roman year subsequently fell into a state of utter confusion. This was due in no small measure to the caprice or dishonesty (5) of the College of Pontiffs, or the Pontifex Maximus, part of whose functions it was to adjust the Lunar year to the Solar by the insertion of the intercalary month, *Mercedonius*.

(1). See the article "Roman Calendar," in Smith's "Dictionary of Greek and Roman Antiquities;" and Ideler, "Handbuch," II. 31. Of the earlier history of the Roman mode of reckoning, our knowledge is very scanty and imperfect. Mommsen (Hist. of Rome, vol. I.) maintains that the Romans, and probably the Italians generally, had at first an independent reckoning of their own, before they came under the influence of Greek civilization. Of the year of *Romulus* we know almost nothing, *with certainty* (Ideler, II. 31).

(2). Histor. lib. I. 19:—"Quem (annum) intercalaribus mensibus interponendis ita dispensavit ut *vigesimo* anno ad metam eandem Solis unde orsi sunt, plenis annorum omnium spatiis, dies congruerent." In translating the word 'vigesimo,' it is necessary to bear in mind that the Romans counted *both* extremes in defining the interval from one point to another. A similar usage we see in the French expressions, *huit jours*, for a *week*, and *quinze jours*, for a *fortnight*.

(3). The reading is uncertain; some codices have *quarto et vigesimo*, instead of *vigesimo*.—*Vid.* Ideler, II. 70.

(4). The origin of this term is uncertain. It has been suggested that it is derived from *merces*, because yearly wages were paid in this month. But, curiously enough, no Latin author uses the word, the term *mensis intercalaris* taking its place. Plutarch writes it Μερκηδόνιος, or Μερκιδῖνος.—*Vid.* Ideler, II. 60.

(5). They lengthened or shortened the year, as it served the ends of their political party, or thwarted those of their opponents; or in order to increase or diminish the profits of the farmers of the public revenues; or for other corrupt purposes. To such an extent did the confusion proceed, that the annual festivals were entirely misplaced, so that, as Suetonius says, "Neque messium feriæ æstati, neque vindemiarum auctumno competerent."

16. Julius Cæsar, as Pontifex Maximus, in his third consulate (U. C. 708, B. C.

46), undertook to reform this disorder, with the aid of Sosigenes, an eminent Alexandrian astronomer and mathematician (1). The task which he had to perform was twofold: namely, first, to correct the errors already existing; and, secondly, to guard against their occurrence in future.

I. First, then, the Kalends of January had receded from their original place in the *Solar* year (the *bruma*, or winter Solstice) almost to the autumnal Equinox. In the year 46 B. C., the winter Solstice for the meridian of Rome fell on the 24th of December (Julian reckoning). Cæsar, it seems, would have assigned the *bruma* to the 1st of January; but, wishing to disturb as little as possible the arrangement of the old *Lunar* year of Numa, he preferred to *date the first year of his reformed Calendar from the new moon next after the bruma;* which new moon fell on the night between the 1st and 2nd of January, B. C. 45. Accordingly, he added to the preceding year (U. C. 708) two other intercalary months, containing 67 days; thus making the number of months in that year 15, and the number of days, 445 (2). Hence this year (B C. 46) was commonly called "the year of confusion" (*annus confusionis*); but far more fitly by Macrobius, *ultimus annus confusionis.* The winter Solstice was assigned to the 25th of December (viii. Kal. Jan.), a day later than the true *bruma,* as given above. In conformity with this, the vernal Equinox was assigned to March 25 (viii. Kal. April.); the summer Solstice, to June 24 (viii. Kal. Quint.); and the autumnal Equinox, to September 24 (viii. Kal. Oct.).

(1). Julius Cæsar, during his abode in the East, had become familiar with the use of the Solar year; and, probably, during his sojourn at Alexandria, had become acquainted with Sosigenes, the astronomer.

(2). These 445 days were thus made up:—The ordinary Lunar year, 355 days; the ordinary intercalary month, Mercedonius, 23 days; two extraordinary intercalary months, interposed between November and December, one of 33, the other of 34 days (67 days) = 445 days (Ideler, II. 121).

17. II. The provision for the future was as follows. The year was made entirely Solar, instead of Luni-Solar (1). The commencement of each new year was fixed, as above stated, to the Kalends (or 1st) of January. The length of the Tropical year was taken to be 365¼ days exactly (2). Every fourth year, beginning with the first year of the new reckoning (B. C. 45), was to be intercalary, consisting of 366 days ($\frac{1}{4} \times 4 = 1$), three consecutive years being 365 days each. The intercalary day was inserted in the place occupied by the intercalary *month* (Mercedonius) of the old Lunar year—namely, between the Festivals of the *Terminalia* and the *Regifugium*, the 23rd and 24th of *February* (vii. and vi. Kal. Mart.). The day thus intercalated was called *Bissexto*, or *ante diem Bissextum* Kal. Mart. (3); and the year of intercalation, *annus bissextilis.* The form *bissextilis* (whence our term *bissextile*) does not occur in any writer prior to Bede. The English term "Leap-year" the dictionaries erroneously explain by saying that every fourth *year leaps* over a day more than a common year. The correct explana-

tion is given in the Prayer Book of 1604: namely, "On every fourth year, the *Sunday Letter* leapeth." There is also, in modern Calendars, a difference between the Civil and Ecclesiastical method of intercalation, which I shall have occasion to notice hereafter (Art. 25).

(1). To the 355 days of Numa's Lunar year he added ten days: two to each of the months of January, Sextilis, and December; and one to each of the months of April, June, September, and November, which before had only 29 days each. In order not to alter the intervals between the festivals of each month, or, as Censorinus expresses it, *ne religiones sui cujusque mensis a loco submoverentur*, he added the new days to the *end* of each month (Ideler, II. 125).

(2). The true length of the year had been more approximately ascertained by Hipparchus, at Rhodes, more than a century before; and Sosigenes must have been well acquainted with it; so that the less correct length of 365¼ days was deliberately preferred, doubtless because it offered a simple and convenient method of intercalation. No provision was made for the future correction of the accumulation of error arising from this cause (*vid. infr.* Art. 72).

(3). The way in which the intercalation took place is shown in the following Table, in reading which it must be remembered that the Romans reckoned the days of the month *backwards*, and *included* the day from which they reckoned:—

Common Year.		Bissextile.	
VII. Kal. Mart., .	23 Feb.	VII. Kal. Mart., .	23 Feb.
		Bissext., ...	24 „
VI., ...	24 „	VI., ...	25 „
V., ...	25 „	V., ...	26 „
IV., ...	26 „	IV., ...	27 „
III., ...	27 „	III., ...	28 „
Prid. Kal., . . .	28 „	Prid. Kal., . . .	29 „
Kal. Mart., . . .	1 March.	Kal. Mart., . . .	1 March.

Thus, in a common year, the 24th of February was counted the *sixth* day before the Kalends of March. The day intercalated (in a leap year) between the 24th and 23rd was called *bissextum* (not *-us*), or *posteriorem*, i. e. the *second*, or *latter*, sixth, counting backwards. Celsus informs us that the Roman Jurists raised the question whether the former (*prior*) or the latter (*posterior*) of the two days which were called *a. d.* VI. *Kal. Mart.*, was to be considered as the *bissextum;* and that they decided for the *posterior;* that is to say, for the day *more remote* from the Kalends of March (Ideler, II. 130, 621).

From what has been said, it would appear that our present distribution of the days of the year among the twelve months is due to Julius Cæsar, and not to Augustus, as some have thought. At all events, one statement—namely, that, in the *Julian* arrangement, February consisted of 29 days in common years, and of 30 in Leap-years—is irreconcileable with the term *bissextum* applied to the intercalary day. Because, if February contained 29 days in a common year, the 24th would not be the 6th, but the 7th day before the Kalends of March. The only change made subsequently to Julius Cæsar seems to have been that of the months *Quintilis* and *Sextilis*, respectively, into Julius and Augustus. Julius' name was introduced into the new Calendar, the year after its publication, being also the year of his death (B. C. 44). It was substituted for Quintilis, the month in which he was born. Similarly, the name Augustus was (by a Senatus Consultum, and a Plebiscitum) substituted for Sextilis (B. C. 8), the same year in which the error in the mode of intercalating was rectified (Art. 18, note 1). Augustus was born in September; but Sextilis was chosen in preference, because some of the most remarkable events in his public life had taken place that month. It must be remembered that, in the ancient Roman reckoning, the year began with *March*, so that, counting inclusively, July was the *fifth* month of the year (Quintilis), and August the *sixth* (Sixtilis).

18. It was Julius Cæsar's intention that the intercalary day should be inserted *peracto quadriennii circuitu*, as Censorinus expresses it. The phrase which Cæsar used was, probably, *quarto quoque anno*, which the Pontiffs, after his death (B. C. 44), understood as meaning anno quarto *incipiente* (not *peracto*), i. e. every *three* years (1). Accordingly, instead of intercalating *nine* days, during the interval from B. C. 45 to B. C. 9, they intercalated *twelve*. When this error was detected, Augustus decreed (B. C. 8) that, in order to drop the three superfluous days, there should be no intercalation for the next twelve years. Accordingly, the years U. C. 749, 753, 757, or B. C. 5, A. D. 1, 4, were reduced to common years. The intercalation was resumed, U. C. 761, or A. D. 8; and the Julian year brought back to its proper track. A similar error was avoided for the future, by engraving on a brazen tablet the correct interpretation of the Julian rule—namely, *quinto quoque incipiente anno* (Ideler, II. 130–133).

It is true that only fragments of the Julian Fasti have come down to us (2), and that the accounts which we have of Augustus' correction are not so perfectly clear as to leave no doubt remaining respecting it; yet, astronomers and chronologers have agreed to accept A. D. 8 (U. C. 761) as a *Leap*-year, and to reckon backwards as well as forwards from it, so as to carry the Julian reckoning into past time as if it had suffered no interruption from the cause just noticed. Consequently, A. D. 45, the initial year of the Julian reckoning is counted a *Leap*-year, as Cæsar most probably intended it to be.

The series of Leap-years, accordingly, in the Julian Chronology, is arranged as follows, the dots indicating Leap-years:—

B. C., ... $\dot{9}876\dot{5}432\dot{1} \parallel 123\dot{4}567\dot{8}9$ . . . . A. D.

that is to say, A. D. 4 is the first Leap-year *after* the Christian Era; B. C. 1 is the first *before* it.

(1). The mistake most probably originated in the peculiar Roman usage, already referred to, of counting the *two* extremes in expressing any interval of time; so that the Pontiffs, not understanding the principle of Cæsar's intercalation, thought that every *fourth* year meant what we should call every *third*.

(2). The most complete account of the Roman *Fasti Sacri* is given by Foggini, in his "Fast. Ann. Roman.," 1779.—*Vid.* Smith's Dictionary, Art. "Fasti." (Ideler, II. 135).

19. Before passing from the Julian year it will be well to notice the very peculiar manner in which the Romans divided their months, and reckoned the days of the months; especially, as this division was continued in the old Church Calendar; and from the very first has ever constituted an authorized part of our own Prayer Book. The following exhibits the divisions and notation of the month of *January*, which will serve as a sample of the rest:—

Days of Month.		Days of Month.	
1	Kalendæ Januariæ, or Januarii.	17	a. d. XVI. Kal. Febr.
2	a. d. IV. Nonas Januarias.	18	a. d. XV. ,,
3	a. d. III. Non. Jan.	19	a. d. XIV. ,,
4	Pridie Non. Jan.	20	a. d. XIII. ,,
5	Non. Jan.	21	a. d. XII. ,,
6	a. d. VIII. Idus Januarias.	22	a. d. XI. ,,
7	a. d. VII. Id. Jan.	23	a. d. X. ,,
8	a. d. VI.	24	a. d. IX. ,,
9	a. d. V.	25	a. d. VIII. ,,
10	a. d. IV.	26	a. d. VII. ,,
11	a. d. III.	27	a. d. VI. ,,
12	Pridie Id. Jan.	28	a. d. V. ,,
13	Idus Januariæ.	29	a. d. IV. ,,
14	a. d. XIX. Kal. Febr.	30	a. d. III. ,,
15	a. d. XVIII. ,,	31	Pridie Kal. Febr.
16	a. d. XVII. ,,		

The rest of the Roman Calendar may be easily completed from this, bearing in mind that in March, May, July, and October, the Nones fall on the 7th, and the Ides on the 15th; whereas in January, and the remaining months, the Nones fall on the 5th and the Ides on the 13th. Hence, in those four months, the 2nd day of the month is denoted by a. d. VI. Non., instead of IV. From the Nones to the Ides (inclusive) eight days elapsed, so that the day after the Nones was always called a. d. VIII. Idus. The *Kalends* were so called, according to Macrobius, from the fact that one of the minores pontifices, whose office it was to look out for the first appearance of the Moon after conjunction, and inform the rex sacrificulus of it, then summoned the plebs (plebe *calata*, i. e. *vocata*) to the Capitol, and announced to them the number of days (5 or 7) from that day (thence called *Calendæ*) to the Nones. The *Nones* seem to have been so called because thence to the Ides (*both* inclusive) *nine* days elapsed. The *Ides*, again, most probably were so called (as Macrobius tells us) from an old Etruscan word *iduare*, meaning *dividere*.—(Ideler, II., 38–45.)

All this seems to confirm the *lunar* origin of the ancient Roman month: the *Kalends* referring to the *New* moon; the *Nones* to the *First quarter;* and the Ides to the *Full* moon. The second half of the month had no division in it, but the whole interval from Full moon to the next New moon was reckoned simply in reference to the latter. The curious custom of counting the days *backward* also falls in quite with this opinion that the Roman year was originally Lunar. For the expressions ante diem IV. Non.; ante diem VIII. Id.; ante diem XIX. Kal. Feb., will then amount to this—it wants so many days to First Quarter, to Full moon, to New moon. It may also be noticed how the peculiarity of the Roman mode of reckoning, before alluded to, viz., *including both* extremes, shows itself in the above notation. The letters a. d. are an abreviation of *ante diem,* and are, in fact, the same as *die.* Thus, when Cicero says, "Scripsi a. d. XVI. Kal. Feb.," it does not mean that he wrote *before* the 17th of January, but *on* that day. The most probable explanation of this curious phrase is, that there is a transposition of *ante;* so that the phrase originally was diem decimum sextum ante Kalendas.

20. The Calendars of the early Christian nations in general were, with a few modifications, essentially based, as to their form and divisions, upon that of Julius Cæsar (1). The eras commonly used by the various nations among whom they dwelt were also those employed in Christian records, during the first five centuries. Of these eras the principal, in the West, were The Building of Rome (U. C.), and the Era of Diocletian. About the year 532 (A. D.), Dionysius Exiguus, a Scythian by birth, and a Roman Abbot, proposed that all Christians should adopt the epoch of the Birth of Christ as their point of departure in counting time and recording dates. As the result of his researches and calculations, he assigned this event to Dec. 25, U. C. 753. This was called the *Vulgar* or *Dionysian Era,* and gradually superseded all others (2). The first year of this era was not, however, made to commence on the day of the Nativity, because it would have been inconvenient to transfer the beginning of the year from its long-established place on January 1st. Dionysius, accordingly, assigned the commencement (or epoch) of the Christian Era to Jan. 1, U. C. 754. It must be carefully observed that A. D. 1 is not *the year* of the Nativity, but the *first* current *year after it;* in other words, it is the year in which Christ *completed* his first year. His birth took place B. C. 1 (3).

(1). The transition from the heathen to the purely Christian Calendar was gradual. There are still extant two Calendars, of the 4th and 5th century, respectively, in which we find heathen and Christian elements combined; *e.g.* the eight *Nundinæ* Letters, along with the *seven* Letters of the Christian *week.* The earliest known *purely* Christian Calendar is a Gothic one of the 4th cent., probably drawn up in

Thrace. It is printed by *Mai*, in the 5th vol. of his Scriptor. vet. nov. Collect. See *Piper*, art. Kalendar, in *Herzog's* Real-Encyclop.

(2). In Italy the Dionysian Era was generally adopted before the close of the 6th century. Its adoption in England was chiefly due to Bede, though a supposed instance is adduced of its use there as early as A. D. 680. The Council of Chelsea (816) decreed that all Bishops should date their Acta from the year of the Incarnation. In Spain, the Christian Era was not uniformly employed in public documents till after the middle of the 14th cent. In the Papal diplomas Mabillon has not found it prior to the time of Leo IX., about the middle of the 11th century. In the Eastern Empire, and in Greece, it was not universally adopted until after the capture of Constantinople (1453). See *Sir H. Nicolas*, Chron. of Hist., p. 4. Ideler, II., 365, *sq.*

(3). As the first century *began* on the midnight with which Jan. 1, A. D. 1, commenced, so it *ended* when, dating from that moment, 100 complete years had elapsed; that is to say, at the midnight between Dec. 31, A. D. 100, and Jan. 1, A. D. 101. Similarly, the second century ended and the third began at the midnight between Dec. 31, A. D. 200 and Jan. 1, A. D. 201. The entire year A. D. 100 belonged to the first century, and the entire year A. D. 200 to the second. In the same way, the first day of the present century was Jan. 1st, 1801; and its last day will be Dec. 31st, 1900.

21. As in counting *forwards* from the Birth of Christ, the year next *after* that event was reckoned as the first of the series A. D., so, *in counting backwards*, the year next *before* that same event ought to have been reckoned as the first of the series B. C.; and the year itself of the Nativity should have been designated 0, so as to preserve the continuity of the whole, unbroken. That is to say, the whole series should have been written thus, ... 3, 2, 1, 0, 1, 2, 3 . . . . , zero being common to both sides. This, accordingly, is the scale adopted by astronomers [1] in denoting the years before Christ. With them the interval between any day in any year A. D., and another day in any year B. C., is expressed by simply adding together the numbers denoting the years. For instance, as the interval between Dec. 25, A. D. 1, and the Nativity year 0, was one year, so the interval between Jan. 1, A. D. 1, and Jan. 1, B. C. 1, is 2 years; and so on. But historians and chronologers have not adopted this (correct) mode of reckoning. With them, the year immediately preceding A. D. 1, is counted as B. C. 1. The result of which is that the *historical* date of any year B. C. exceeds by 1 the *astronomical* date. So that in calculating, from historical dates, the interval from any day in any year B. C. to the corresponding day in any year A. D., 1 must be deducted from the sum of the numbers expressing the years, *e. g.* the interval from B. C. 10 to A. D. 10 is (20–1) 19 years. It is to be borne in mind, also, that, in calculating such intervals, the years both before and after Christ are *current*, not *elapsed*. For example, if we seek the interval between the epochs of the era of the Seleucidæ and that of the Arabians (Hegirah), the former of which dates Oct. 1, B. C. 312, and the latter July 15, A. D. 622, we have 311 complete years and 3 months *before*, and 621 complete years and 6½ months *after* Christ; the sum of which is 932 complete years, 9½ months [2].

(1). This mode of reckoning was first proposed and employed by the astronomer J. Cassini. *Vid.* Ideler, I. 76.

(2). To convert this interval into days, we must multiply 932 by 365, and add the 233 Leap-years; and also the number of days (92) in the last three months of B. C. 312, together with the number (181) in the first five months of A. D. 622, *plus* the 14 elapsed days of July. Hence we get, 340180 + 233 + 92 + 181 + 14 = 340700 days.

22. We may now proceed to show *How the Sundays in any Julian year are determined.*

In the first place, the entire year of 365 days was subdivided into *weeks* (1), or groups of seven days, beginning with the 1st of January. To the seven days of each group were attached the first seven letters of the alphabet, in their order, A, B, C, D, E, F, G; the series being repeated fifty-two times, with one day and one letter over. This mode of denoting the seven days of the week was obviously suggested by the *Nundinæ*, or *eight*-day week of the Julian Calendar, in which this division of time, originating probably with the Etruscans, was continued (2). The *seven* letters are called the CALENDAR LETTERS, which must not be confounded with the *Sunday Letters*, as they sometimes have been. In affixing these letters to the days of the year, it is important to observe that no notice was taken of Leap-year, when February has 29 days; in other words, February 29 was passed over, and had no letter affixed to it. The letter D, following C, which belongs to February 28, was affixed to March 1. This arrangement of the Letters is exhibited in the Prayer Book, in the Monthly Calendar, where the series A, b, c, d, e, f, is repeated throughout the year, beginning with January 1. A alone is printed as a capital letter, to indicate that, when the first of January falls on Sunday, all the days throughout the year to which A is affixed are Sundays; the small letters indicating the successive week days in their order. When the first Sunday in the year falls on any other of the first seven days of January, the Letter corresponding to that day is the Sunday Letter of that year; and the other six (smaller) Letters designate the corresponding week days. For instance, if the first Sunday fall on January 2, B is the Sunday Letter of the year, and so on; and, generally, as the first Sunday is constantly changing its position, "THE LETTER WHICH STANDS OPPOSITE *the first Sunday in any year is called the* DOMINICAL *or* SUNDAY LETTER *of that year.*" In the Church Almanacs this letter is usually printed in red ink, for distinction's sake. It is obvious, as already intimated, that what has just been said of *Sunday* holds equally of every other day of the week. For instance, if January 1 fall on Monday, then a will be the *Calendar* Letter of all the Mondays in the year, as B will be the *Sunday* Letter. All the Tuesdays will have c as their *Calendar* Letter; or, in other words, all the days of the year, to which c is affixed, will be Tuesdays, and so on of the rest.

(1). The Week, or seven-day division of time, is found amongst the most widely different nations of the Earth. In the Eastern world, it was employed from a remote antiquity by the Chinese, Hindoos, Assyrians, and Egyptians; while in America it was in use among the ancient Peruvians. This clearly indicates that it was a *national*, not a mere conventional division. It was, most probably, the fourth part of the synodic month (29½ days), each quarter being marked by conspicuous difference of phase. The fourth part consists of 7⅜ days, of which the next lower integer was taken (Ideler, I. 60, 80). Among the Shemitic nations, the week was in ordinary use, at least as early as the time of Moses. The Greeks and Romans became acquainted with it chiefly, it would seem, through the Jews dispersed among them. The Greeks never adopted it into their Calendar, nor did the Romans, until after the establishment of Christianity by Constantine. It was first recognised by him, and finally legalised in the Theodosian Code, early in the 5th century. The word *septimana* (week) occurs for the first time in this Code (*Dominico qui septimanæ totius primus est dies*). According to Dion Cassius, the ancient Ægyptians derived the number and order of the seven days from the seven most conspicuous celestial objects—namely, the Sun and Moon, and the five Planets then known; Saturn, as the most remote from the Earth, being taken as the first of the series (lib. xxxvii. c. 17). The Roman names of the week-days—viz. Dies Saturni, Solis, Lunæ, Martis, Mercurii, Jovis, Veneris—are the ultimate source from which our modern names are derived. The Pagan Saxons substituted for the names of the days dedicated to the heavenly bodies the names of the corresponding divinities in the Scandinavian Mythology. Hence our English names of the week-days are, partly of Roman, partly of Scandinavian descent. To the former belong Sunday, Monday, and Saturday; to the latter, the remaining four days. In all our legislative and judiciary acts and documents, the Latin names of the days of the week are still employed.

(2). The days of the year, beginning with January 1, were subdivided into successive groups of *eight* days, with the letters from A-H attached to the several days. In other words, the Roman week consisted of eight days. The farmers worked seven days, and on the eighth (*nono quoque die*) went into the city to *market*, and to acquaint themselves with public affiairs. Between two successive nundinæ there were, accordingly, seven ordinary days. In their peculiar form of reckoning (before referred to), the Romans added the *two* extremes to the seven intermediate days. Thus, the successive groups were regarded as recurring *nono quoque die*, and were accordingly designated *novendinæ*, or *nundinæ*. This mode of reckoning continued until the time of Constantine. He transferred the market days to the Sundays. And thus the seven-day week, which before was peculiar to the Christians, was extended to the whole nation.

23. The following Table exhibits at one view the CALENDAR LETTERS corresponding to all the days of the year, and also the year-number of each day, reckoned from January 1. The Calendar Letters are all printed in the same character, as is necessary in a *general* Table, applicable to all years. The Table is constructed for a *Common* year, but also applies, with the necessary modifications, to a Leap year.

TABLE SHOWING THE CALENDAR LETTER AND YEAR NUMBER OF EVERY DAY IN THE YEAR.

Days of Month.	Jan.		Feb.		Mar.		Apr.		May.		June.		July.		Aug.		Sept.		Oct.		Nov.		Dec.		Days of Month.
	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	Cal. Letter.	Day of Year.	
1	A	1	D	32	D	60	G	91	B	121	E	152	G	182	C	213	F	244	A	274	D	305	F	335	1
2	B	2	E	33	E	61	A	92	C	122	F	153	A	183	D	214	G	245	B	275	E	306	G	336	2
3	C	3	F	34	F	62	B	93	D	123	G	154	B	184	E	215	A	246	C	276	F	307	A	337	3
4	D	4	G	35	G	63	C	94	E	124	A	155	C	185	F	216	B	247	D	277	G	308	B	338	4
5	E	5	A	36	A	64	D	95	F	125	B	156	D	186	G	217	C	248	E	278	A	309	C	339	5
6	F	6	B	37	B	65	E	96	G	126	C	157	E	187	A	218	D	249	F	279	B	310	D	340	6
7	G	7	C	38	C	66	F	97	A	127	D	158	F	188	B	219	E	250	G	280	C	311	E	341	7
8	A	8	D	39	D	67	G	98	B	128	E	159	G	189	C	220	F	251	A	281	D	312	F	342	8
9	B	9	E	40	E	68	A	99	C	129	F	160	A	190	D	221	G	252	B	282	E	313	G	343	9
10	C	10	F	41	F	69	B	100	D	130	G	161	B	191	E	222	A	253	C	283	F	314	A	344	10
11	D	11	G	42	G	70	C	101	E	131	A	162	C	192	F	223	B	254	D	284	G	315	B	345	11
12	E	12	A	43	A	71	D	102	F	132	B	163	D	193	G	224	C	255	E	285	A	316	C	346	12
13	F	13	B	44	B	72	E	103	G	133	C	164	E	194	A	225	D	256	F	286	B	317	D	347	13
14	G	14	C	45	C	73	F	104	A	134	D	165	F	195	B	226	E	257	G	287	C	318	E	348	14
15	A	15	D	46	D	74	G	105	B	135	E	166	G	196	C	227	F	258	A	288	D	319	F	349	15
16	B	16	E	47	F	75	A	106	C	136	F	167	A	197	D	228	G	259	B	289	E	320	G	350	16
17	C	17	F	48	F	76	B	107	D	137	G	168	B	198	E	229	A	260	C	290	F	321	A	351	17
18	D	18	G	49	G	77	C	108	E	138	A	169	C	199	F	230	B	261	D	291	G	322	B	352	18
19	E	19	A	50	A	78	D	109	F	139	B	170	D	200	G	231	C	262	E	292	A	323	C	353	19
20	F	20	B	51	B	79	E	110	G	140	C	171	E	201	A	232	D	263	F	293	B	324	D	354	20
21	G	21	C	52	C	80	F	111	A	141	D	172	F	202	B	233	E	264	G	294	C	325	E	355	21
22	A	22	D	53	D	81	G	112	B	142	E	173	G	203	C	234	F	265	A	295	D	326	F	356	22
23	B	23	E	54	E	82	A	113	C	143	F	174	A	204	D	235	G	266	B	296	E	327	G	357	23
24	C	24	F	55	F	83	B	114	D	144	G	175	B	205	E	236	A	267	C	297	F	328	A	358	24
25	D	25	G	56	G	84	C	115	E	145	A	176	C	206	F	237	B	268	D	298	G	329	B	359	25
26	E	26	A	57	A	85	D	116	F	146	B	177	D	207	G	238	C	269	E	299	A	330	C	360	26
27	F	27	B	58	B	86	E	117	G	147	C	178	E	208	A	239	D	270	F	300	B	331	D	361	27
28	G	28	C	59	C	87	F	118	A	148	D	179	F	209	B	240	E	271	G	301	C	332	E	362	28
29	A	29		[60]	D	88	G	119	B	149	E	180	G	210	C	241	F	272	A	302	D	333	F	363	29
30	B	30			E	89	A	120	C	150	F	181	A	211	D	242	G	273	B	303	E	334	G	364	30
31	C	31			F	90			D	151			B	212	E	243			C	304			A	365	31

The use of this Table is obvious. If the 1st of January (for example) be *Sunday*, then every A in the Table will denote Sunday; every B, Monday, and so on. If, again, we know the Letter belonging to any given day of the week, in any given year, the Table shows at once the day of the week corresponding to any given day of any month that year: *e. g.*, if A be the *Sunday* Letter of any year (1871), and we wish to know the day of the week on which the 10th of October falls that year, we see that C is the Calendar Letter of that day; and, therefore, the required week-day is Tuesday; or, if A be the *Monday* Letter of any year, then the 10th of October (C) will be Wednesday. And, *v. v.*, if we know any week-day Letter, we can find at once the days of each month corresponding to it, or any other week-day: *e. g.*, if A be the *Sunday* Letter and we wish to know what days of July will be Wednesdays, the Wednesday Letter will be D, and all the days to which D is prefixed—viz., 5, 12, 19, 26—will be Wednesdays. The inspection of the Table also shows us that, in a *Common* year, the 31st of December has the same Letter (A) as the 1st of January; in other words, *the year ends on the same day of the week that it began:* which is also obvious from this, that 365 days contain 52 complete weeks *plus* 1 day (1).

We see also that January 1 and October 1 have the same Letter (A), and, therefore, these months commence on the same day of the week. So do February, March, and November (Letter D); so, also, April and July (G); September and December (F). This Table also shows at once the number of days from any day of the year to any other day. For example, from February 1 to November 28: February 1 is the 32nd day of the year; and November 28 is the 332nd; hence number required is 332 − 32 = 300. This *includes* one of the extremes—either February 1, or November 28. If *both* extremes be included, the number is 301; if *neither*, 299.

It is useful to remember the Calendar Letters of the *first* day of each month, because this will suffice to determine, without the Table, the Calendar Letter of *every* day in each month. For this purpose, the following doggrel couplet has been invented.

Jan.	Feb.	March.	April.	May.	June.
At	Dover	Dwell	George	Brown	Esquire
July.	August.	Sept.	Oct.	Nov.	Dec.
Good	Christopher	Finch	And	David	Friar.

A more ancient and graver couplet is given by Clavius:—

Astra	Dabit	Dominus,	Gratisque	Beabit	Egenos;
Gratia	Christicolæ	Feret	Aurea	Dona	Fideli.

(1). As every month consists of four complete weeks, and (with the exception of February) two, or three, days over, it is obvious that, if the initial day of any month be given, the initial day of any other month can be found, without the Table, by multiplying the number of *intermediate* months (omitting February) by 2, adding 1 for each month of 31 days, and dividing the resulting sum by 7. The *remainder* will be the number of days of the week which the required day is distant from the given one: *e. g.*, if March 1 fall on Tuesday, required the day of the week on which September 1 falls. Here we have six intermediate months, four of which contain 31 days. Therefore, $\frac{6 \times 2 + 4}{7} = \frac{16}{7}$; leaving the remainder 2: hence, September 1 falls 2 days later in the week than March 1; *i. e.*, on Thursday, as the Table shows. Similarly, if January 1 fall on Sunday, October 1 will also fall on Sunday, because there are 8 intermediate months, omitting February, 5 of which contain 31 days; and $\frac{8 \times 2 + 5}{7}$ leaves *no* remainder: in other words, October, in a Common year, begins the same day of the week that January does (see the Table). In a Leap-year, 1 day must be added for February, and the remainder will be 1, showing that in a Leap-year October 1 falls one day of the week later than January 1.

24. Hitherto we have been dealing with the *Common* year, of 365 days. We have now to consider what changes take place in the case of *Leap*-year.

In the first place, it is obvious that, by the addition of a 29th day to February, the *year-number* of each subsequent day must be increased by 1—thus, March 1 becomes the 61st (instead of the 60th) day of the year; and so on to the end—and that the year, therefore, ends, not on the *same* week-day on which it commenced, but on the next: *e.g.*, if the year begin on Sunday, it will end on Monday.

Secondly, the insertion of the intercalary day will cause a change in the Sunday Letter subsequent to February 29. Because in the week in which the intercalary day occurs, *two days have the same Letter*, and, therefore, from the preceding Sunday to the next following Sunday, there will be only *six* Letters reckoned: consequently, the Letter of the latter Sunday must be one less than the Letter of the former Sunday. Thus, for example, the Sunday Letter of 1872 (Leap-year) was G. Consequently February 25 (G) was Sunday; 28th (C), Wednesday; 29th (to which no Letter is attached, or else that of March 1), Thursday; March 1 (D), Friday; March 3 (F), Sunday. Therefore, F (instead of G) denotes *that* Sunday, and all the remaining Sundays of the year. And, generally, in a Leap-year, the Sunday Letter, after the 29th of February, is the Letter immediately *preceding* the Sunday Letter at the beginning of the year, and up to February 29. Or, to state the same result in another form, the effect of the intercalation is to push up all the week-days after the 28th of February one place, while the series of Calendar Letters is left untouched. *A Leap-year, therefore, has two Sunday Letters, the first applicable to the first two months of the year, and the other to the last ten; the second Letter being, in the order of the letters of the alphabet, one behind the first; e. g.,* the Sunday Letters of 1872 were G, F.

Hence it also follows, that, as a *Common* year ends with the same day of the week with which it began, the first seven days of the year next after a Common year fall upon the week-days which immediately succeed those on which they fell the preceding year; that is to say, the Sunday Letter of the following year is *one behind* that of the preceding. The year following a *Leap-year* has, for its Sunday Letter, a Letter *two behind* that with which the Leap-year *commenced: e. g.*, the Sunday Letter of 1873 was E, and of 1874, D. The Sunday Letters of 1872 were G, F, that of 1873 being E.

25. The Church of Rome has adhered to the ancient Ecclesiastical Calendar, in placing the intercalary day between the 23rd and 24th of February, as the old Julian Calendar did (Art. 17); thus making, in fact, two 24ths, as in the Julian Calendar there were two vi. Kalend. Hence the change of Sunday Letter, in Leap-year, takes place in the Roman Calendar after the 24th of February; not, as with us, after the 29th. The note in the Roman Breviary and Missal is "In anno bissextili, Februarius est dierum 29; et Festum S. Mathiæ celebratur 25 Februarii, et *bis dicitur Sexto Calendas*, id est, die 24 et die 25, et Litera Dominicalis quæ assumpta fuit in mense Januario mutatur in præcedentem: ut si in Januario Litera Dominicalis fuerit A, mutatur in præcedentem, quæ est g, et litera f bis servit, 24, 25." The same rule is laid down in the Sarum Missal: "Si *Bissextus* fuerit, quartâ die a Cathedra S. Petri (February 22) fiat Festum S. Mathiæ." In our first Prayer Books, the ancient rule was followed, or intended to be followed, as to the mode of intercalation, and the change of St. Matthias' day to the 25th in Leap-years. In the Church Calendar, the place of the *Regifugium* was taken by St. Matthias' day. But at the final revision, in 1662, the ancient practice was given up, and the Civil mode of intercalation—namely, making the 29th day of February the intercalary day—was adopted. The Revisers of 1662 were, probably, induced to make this change in order to put an end to the doubts and mistakes which had arisen respecting the proper day on which the Festival of St. Matthias should be kept (1).

The following Table shows the arrangement respecting Leap-year in the Roman and the English (or Civil) Calendar, respectively. The asterisks indicate the intercalary days. Compare the corresponding portion of the old Julian Calendar, Note 3, Art. 17.

Common Year.		Leap-year, Roman Calendar.		Leap-year, Civil Calendar.	
22 Feb.,	D	22 Feb.,	D	22 Feb.,	D
23 ,,	E	23 ,,	E	23 ,,	E
24 ,,	F	24 ,,	F *bis.**	24 ,,	F S. Matthias.
25 ,,	G	25 ,,	F S. Matthias.	25 ,,	G
26 ,,	A	26 ,,	G	26 ,,	A
27 ,,	B	27 ,,	A	27 ,,	B
28 ,,	C	28 ,,	B	28 ,,	C
		29 ,,	C	29 ,,	D *bis.**
1 Mar.,	D	1 Mar.,	D	1 Mar.,	D

Thus, in both modes of reckoning, and whether the year be Common or Bissextile, March 1 has always the Letter D. In the Roman Calendar, the intercalation does not disturb the arrangement of the Calendar Letters, except from February 26 to February 28. In the Civil Calendar (that of the Anglican Church) there is no displacement at all of the Letters. February 29 merely takes the same Letter (D) as the following day, March 1.

(1). The original compilers of our Prayer Book (1549) so far changed the ancient usage of the Church, as regards the intercalary day, that they placed it between the 24th and 25th of February, instead of between the 23rd and 24th. In the "Order of how the rest of Holy Scripture is appointed to be read," they laid down the following rule:—"This is also to be noted, concerning the Leap-years, that the xxv. day of February, *which in Leap-year is counted for two days*, shall on those two days alter neither Psalm nor Lesson; but the same Psalms and Lessons, which be said the first day, shall serve also the second day." The same rule was continued in the two succeeding Prayer Books, of Edward VI. and Elizabeth (1552, 1559). Dr. Nicholls (Commentary on the Book of Common Prayer) is of opinion that the compilers of our Prayer Book fell into an error respecting the proper day of intercalation, making two 25th days, instead of the old and correct two 24ths. But Wheatly considers it highly improbable that so many learned men should have been ignorant both of the Rubrics and practice of the old English Church. He, therefore, holds that the alteration was made designedly, in order that there might be no confusion in the observance of *St. Matthias' day* (Feb. 24), but that it should be always kept on the 24th, in Leap-year, as well as in a Common year; and not on the 25th in Leap-years, as the Breviary and Missal directed. But if this were the reason, it is strange that in the Calendar of that Prayer Book, as well as in that of Elizabeth and Edward's Books, St. Matthias' day is affixed to February 25. In King James' Prayer Book (1604), not in Elizabeth's, as Wheatly says, the older practice, with regard to *intercalation*, was restored, and the Rule was thus altered:—"When the year of our Lord may be divided into four even parts, which is every fourth year, the Sunday Letter leapeth; and that year the Psalms and Lessons which serve for the xxiii. day of February shall be read again on the day following, except it be Sunday, which hath proper Lessons of the Old Testament appointed." It is plain from this that it was intended to restore the intercalary day to its former place between the 23rd and 24th, and it is equally clear that it assumes the existence in the English Church of the Roman rule, that, in a Leap-year, St. Matthias' day should be kept on the 25th; because, if the Lessons for the 23rd were also to be read on the 24th, in Leap-years, then that day could not be St. Matthias', which had its own proper first Lessons different from those of the 23rd.

The Reviewers of 1662 wholly omitted the Rule of the preceding Prayer Books relative to the repetition of the Psalms and Lessons in Leap-year, and adopted the Civil method of intercalation—viz., adding a 29th day to February, making it the intercalated day, and appointing Lessons for it: the first Lessons in their course, the second Lessons out of course (Matt. vii. and Rom. xii.). They moreover fixed the Feast of St. Matthias permanently to the 24th of February. It may be worth adding that, by an old statute (21 Henry III., A. D. 1236), it is enacted that the place of the intercalary day shall be that which it occupies in the old Church reckoning.

For several years after the Revision of 1622, some of the Almanac makers still adhered to the custom of placing St. Matthias' day in Leap-year on the 25th; which led to some diversity of usage in the Church: to put an end to which, Archbishop Sancroft (who was himself one of the Revisers, and was principally concerned with the Calendar revision) published (1683) an Injunction, requiring "all Parsons, Vicars, and Curates, to take notice that the Feast of St. Matthias is to be celebrated, not upon the 25th of February, but on the 24th of February, for ever, whether it be Leap-year or not, as the Calendar in the Liturgy, confirmed by the Act of Uniformity, appoints and enjoins." Thus the uncertainty about St. Matthias' day was finally removed, and our Church observes it, in Leap-year, on the day before the Church of Rome does. Dr. Nicholls, indeed, denies the authority of Archbishop Sancroft's Injunction, and tries to maintain that the Caroline Revisers did not mean to substitute the Civil for the Ecclesiastical method of intercalation; nor to deviate from the ancient usage of the Church by observing St. Matthias' day on the 24th instead of the 25th.

26. From what has been said (§ 21) it is plain that, if we know the Sunday Letter for any given year in the old Julian Calendar—that is to say, the Julian Calendar before the Reformation of Gregory XIII., in 1582—we can find those of all the preceding and subsequent years. If, for example, we know that A. D. 1000 (a Leap-year) had for its Sunday Letters G F, we also know that the Letter of the preceding year, 999, was A, of 998, B; and so on, until we reach 972, when we arrive at G F again. Similarly, counting forwards from A. D. 1000, we get E for 1001, D for 1002, and so on, until we reach 1028, when we arrive at G F again. And it further appears that in each of these two periods of 28 years—namely, 972–1000, and 1000–1028—the Sunday Letters succeed each other in exactly the same order. Nor is this peculiar to those two periods. It will be found to apply to all the preceding periods of 28 years, up to A. D. 20, and to all the subsequent periods down to 1560, in the Roman reckoning, and to 1748 in our own. And the same, of course, holds true if we start from any other year of which the Sunday Letter, or Letters, is known. In all cases, the Sunday Letters will be found to recur in cycles of 28, their order of succession in each cycle being identical.

This remarkable cycle is called the SOLAR CYCLE, or, more properly, the CYCLE OF THE DOMINICAL LETTERS (Cyclus Solaris; *κύκλος*, or *ὀκτωκαιεικοσαετηρὶς, τοῦ ἡλίου*). It seems to have been made use of about the time of the Council of Nice. It is an immediate consequence of the Julian intercalation of a day every fourth year. If *all* years consisted of 365 days, the Sunday, and all the other Calendar, Letters would recur in the same order at the end of every *seven* years; because, in passing from one Common year to the next, the Sunday Letter retrogrades one place: consequently, at the end of seven consecutive Common years, it would come again to the place from which it started. But the insertion of a day every fourth year interrupts this regular order of regression. Every Leap-year causes the Sunday Letter to recede two places; therefore, in four years it retrogrades *five* places, in other words, every fourth year, the regular rate of regression is increased by one place; therefore, in seven times four years, there will be seven such additions; that is to say, at the end of 28 years the displacements arising from the Leap-years will be all adjusted, because the total number of regressions will be 35—an exact multiple of 7. If, then, a Table of Sunday Letters were constructed for *any* 28 years of the Julian Calendar, it would show the Sunday Letter (or Letters) for every corresponding year of every cycle down to the *Change of Style*, when the Julian Calendar was corrected.

27. In constructing such a Table, any year may be taken at pleasure, as the starting point of the cycle. The starting point originally assumed was defined by this condition—That it should be *a Leap-year*, in which the first of January was Monday, and, therefore, the Sunday Letters G F. The following Table exhibits the resulting *correspondence between the years of the cycle and the Sunday Letters;* the asterisks marking the Leap-years:—

TABLE SHOWING THE RELATION BETWEEN THE SUCCESSIVE YEARS OF THE SOLAR CYCLE AND THE SUNDAY LETTERS.

Years of the Cycle.	Sunday Letter.	Years of the Cycle.	Sunday Letter.
* 1	GF	15	C
2	E	16	B
3	D	*17	AG
4	C	18	F
* 5	BA	19	E
6	G	20	D
7	F	*21	CB
8	E	22	A
* 9	DC	23	G
10	B	24	F
11	A	*25	ED
12	G	26	C
*13	FE	27	B
14	D	28	A

This cycle may be applied to *any* era, of which the Sunday Letter of the initial year, or epoch, is known. Dionysius Exiguus calculated that the Nativity of our Lord took place on *Saturday*, December 25, B. C. 1. Consequently, January 1, A. D. 1, also fell on Saturday; therefore *the Sunday Letter of* A. D. 1 *was B:* A. D. 1 might, accordingly, begin with any of the *three* years of the above cycle which have B for the corresponding Sunday Letter. He choose the 10th year of the cycle, as being the first with that Letter. Consequently, *the Christian Era began on the* 10*th year of the Solar Cycle.* Hence the following ***Rule for finding the Sunday Letter corresponding to any year of our Lord:***—

To the numeral of the given year A. D. *add* 9, *and divide the sum by* 28. *If there be no remainder, the year of the cycle is* 28, *and the required Letter is, by the Table,* A. *If there be a remainder, the Letter or Letters opposite that number will be the required Letter or Letters.* For instance, if the remainder be 1, it will be the first year of the cycle, and the Sunday Letters will be G F; if the remainder be 2, it will be the second year of the cycle, and the Sunday Letter will be E; and so on.

EXAMPLE.—Required the Sunday Letters of A. D. 1000 (Leap-year). Here we have $\frac{1000 + 9}{28} = 36$, with a remainder 1; showing that 36 complete cycles had then elapsed, and that the first year of the 37th cycle was current; and that, therefore, the Sunday Letters are G F.

The Rule may be *generally* expressed by the following convenient formula, where x denotes the number of the year A. D.; r the *remainder* after dividing $x + 9$ by 28; and L the Sunday Letter (or Letters) corresponding to x:—

$$L = \left(\frac{x + 9}{28}\right)_r. \tag{1}$$

The Letter (or Letters) corresponding to the numerical remainder will be found in the Table, as before.

We have just seen (Art. 27) that the epoch, or starting point, of the Solar Cycle is B. C. 9. Consequently, the first cycle ended A. D. 19; and the second cycle began A. D. 20: and so on, continuously.

28. The Table (Art. 27) will also show the Sunday Letter corresponding to any year *Before Christ.* Not, of course, that there were *Sundays* then; but it may still be desirable to know the *Seventh-day Letter*, reckoning backwards from the first Sunday of the Christian Era. The following is the Rule for finding it, the reason of which is obvious, from what has been said above:—

Take the multiple of 28, *which, when added to* 9, *will be the epoch of the cycle next before* *the given year* B. C. *Subtract the latter from the former, and to the difference add* 1. *This will give the year of the cycle, and the corresponding Letter (or Letters) in the Table will be the Sunday Letter (or Letters) sought.* Formula (1) becomes in this case

$$L' = \left(\frac{n\,28 + 9 - x'}{28}\right)_r; \qquad (2)$$

where L' denotes the Sunday Letter (or Letters) corresponding to year-number B. C. (x'), and n the multiple of 28 which will make $n.28 + 9$ greater than x'.

Ex. 1. Find the Sunday Letter of B. C. 200.

$$\begin{array}{rrl} \text{Here we have } 7 \times 28 = & 196 & \\ & +\,9 & \\ \hline \text{B. C.} & 205 & = \text{1st year of a cycle.} \\ & 200 & \\ \hline & 5 & \text{years of the cycle } \textit{elapsed}. \end{array}$$

Therefore 200 is the 6th year of that cycle; and by the Table the Sunday Letter is G.

Ex. 2. Find the Sunday Letters of B. C. 165.

$$\begin{array}{rrl} \text{Here} \quad 6 \times 28 = & 168 & \\ & +\,9 & \\ \hline \text{B. C.} & 177 & = \text{1st year of a cycle.} \\ & 165 & \\ \hline & 12 & \text{years of the cycle elapsed.} \end{array}$$

Therefore B. C. 165 is the 13th year of that cycle; and by the Table the Sunday Letters are F E.

29. Instead of the Table (Art. 27), we may use the following Table, which is, in fact, the same as the former, only that it begins with A. D. 1, whose Sunday Letter is B:—

Solar Cycle, . . .	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Sunday Letter, . .	B	A	G	FE	D	C	B	AG	F	E	D	CB	A	G
Solar Cycle, . . .	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Sunday Letter, . .	F	ED	C	B	A	GF	E	D	C	BA	G	F	E	DC

This Table will determine the Sunday Letter for *any* given year A. D. in the Julian reckoning, or Old Style. The following is the Rule:—

Divide the given year by 28; the remainder will then be the current year of the cycle. The Letter (or Letters) corresponding to this remainder will be the Sunday Letter (or Letters) required.

Ex. 1. A. D. 1066. Here we have

$$\left(\frac{1066}{28}\right)_r = 2 = \text{A}.$$

The *quotient* is 38, showing that 38 complete cycles had elapsed.

Ex. 2. A. D. 1420. Here we have

$$\left(\frac{1420}{28}\right)_r = 20 = \text{GF}.$$

The *quotient* is 50; and, therefore, 50 complete cycles had elapsed.

A similar Table will, of course, show the Sunday Letters for all the years B. C. The Sunday Letters of B. C. 1 are CD; therefore the series for the first cycle, reckoning backwards from B. C. 1, is as follows:—

Solar Cycle, . . .	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Sunday Letter, . .	CD	E	F	G	AB	C	D	E	FG	A	B	C	DE	F
Solar Cycle, . . .	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Sunday Letter, . .	G	A	BC	D	E	F	GA	B	C	D	EF	G	A	B

The Rule is the same as that for years A. D. It may be observed, though, that in a Leap-year the *second* of the Letters is the Sunday Letter for the early part of the year, the *first* for the later part.

Ex. 1. B. C. 1066. Here we have

$$\left(\frac{1066}{28}\right)_r = 2 = \text{E}.$$

Ex. 2. B. C. 1420. Here we have

$$\left(\frac{1420}{28}\right)_r = 20 = \text{F}.$$

On comparing the two Tables in this Article, we see that, while for the years A. D. the series of Letters proceeds in a retrograde order, for the years B. C. it proceeds in a direct order.

30. The Tables which have just been given are sufficient to find the Sunday Letters for any year B. C., or A. D., *Old Style*. The only arithmetical process required is the

division of the given year by 28. But Tables have been constructed by means of which even this easy process is rendered unnecessary, and the Sunday Letter found by a simple inspection. Several such Tables have been drawn up. But, while they differ in some minor details, they are all constructed on the same general plan—namely, a cycle of seven centuries with the Solar Cycle of 28 years. I shall first exhibit the more usual form of these Tables, and then point out the principle of their construction.

TABLE I.

To find, by Simple Inspection, the Sunday Letter (or Letters) for any Year Before Christ.

The Years of each Century B. C.				Centuries B. C.: that is, the Centurial Figures, when the two last Figures of the Date have been dropped.						
				0 7 14 21 28 35 $\left(\frac{X}{7}\right)_r = 0$	1 8 15 22 29 36 $\left(\frac{X}{7}\right)_r = 1$	2 9 16 23 30 37 $\left(\frac{X}{7}\right)_r = 2$	3 10 17 24 31 88 $\left(\frac{X}{7}\right)_r = 3$	4 11 18 25 32 39 $\left(\frac{X}{7}\right)_r = 4$	5 12 19 26 33 40 $\left(\frac{X}{7}\right)_r = 5$	6 13 20 27 34 41 $\left(\frac{X}{7}\right)_r = 6$*
01	29	57	85	CD	BC	AB	GA	FG	EF	DE
02	30	58	86	E	D	C	B	A	G	F
03	31	59	87	F	E	D	C	B	A	G
04	32	60	88	G	F	E	D	C	B	A
05	33	61	89	AB	GA	FG	EF	DE	CD	BC
06	34	62	90	C	B	A	G	F	E	D
07	35	63	91	D	C	B	A	G	F	E
08	36	64	92	E	D	C	B	A	G	F
09	37	65	93	FG	EF	DE	CD	BC	AB	GA
10	38	66	94	A	G	F	E	D	C	B
11	39	67	95	B	A	G	F	E	D	C
12	40	68	96	C	B	A	G	F	E	D
13	41	69	97	DE	CD	BC	AB	GA	FG	EF
14	42	70	98	F	E	D	C	B	A	G
15	43	71	99	G	F	E	D	C	B	A
16	44	72	100	A	G	F	E	D	C	B
17	45	73		BC	AB	GA	FG	EF	DE	CD
18	46	74		D	C	B	A	G	F	E
19	47	75		E	D	C	B	A	G	F
20	48	76		F	E	D	C	B	A	G
21	49	77		GA	FG	EF	DE	CD	BC	AB
22	50	78		B	A	G	F	E	D	C
23	51	79		C	B	A	G	F	E	D
24	52	80		D	C	B	A	G	F	E
25	53	81		EF	DE	CD	BC	AB	GA	FG
26	54	82		G	F	E	D	C	B	A
27	55	83		A	G	F	E	D	C	B
28	56	84		B	A	G	F	E	D	C

* X stands, generally, for the centurial figures.

TABLE II.

To find the Sunday Letter (or Letters), by Simple Inspection, for any Year A. D., Old Style.

The Years of each Century A. D.				Centuries A. D.: that is, the Centurial Figures, when the two last Figures of the Date have been dropped.						
				0 7 14 21 28 35 $\left(\frac{X}{7}\right)_r = 0$	1 8 15 22 29 36 $\left(\frac{X}{7}\right)_r = 1$	2 9 16 23 30 37 $\left(\frac{X}{7}\right)_r = 2$	3 10 17 24 31 38 $\left(\frac{X}{7}\right)_r = 3$	4 11 18 25 32 39 $\left(\frac{X}{7}\right)_r = 4$	5 12 19 26 33 40 $\left(\frac{X}{7}\right)_r = 5$	6 13 20 27 34 41 $\left(\frac{X}{7}\right)_r = 6$
00				DC	ED	FE	GF	AG	BA	CB
01	29	57	85	B	C	D	E	F	G	A
02	30	58	86	A	B	C	D	E	F	G
03	31	59	87	G	A	B	C	D	E	F
04	32	60	88	FE	GF	AG	BA	CB	DC	ED
05	33	61	89	D	E	F	G	A	B	C
06	34	62	90	C	D	E	F	G	A	B
07	35	63	91	B	C	D	E	F	G	A
08	36	64	92	AG	BA	CB	DC	ED	FE	GF
09	37	65	93	F	G	A	B	C	D	E
10	38	66	94	E	F	G	A	B	C	D
11	39	67	95	D	E	F	G	A	B	C
12	40	68	96	CB	DC	ED	FE	GF	AG	BA
13	41	69	97	A	B	C	D	E	F	G
14	42	70	98	G	A	B	C	D	E	F
15	43	71	99	F	G	A	B	C	D	E
16	44	72		ED	FE	GF	AG	BA	CB	DC
17	45	73		C	D	E	F	G	A	B
18	46	74		B	C	D	E	F	G	A
19	47	75		A	B	C	D	E	F	G
20	48	76		GF	AG	BA	CB	DC	ED	FE
21	49	77		E	F	G	A	B	C	D
22	50	78		D	E	F	G	A	B	C
23	51	79		C	D	E	F	G	A	B
24	52	80		BA	CB	DC	ED	FE	GF	AG
25	53	81		G	A	B	C	D	E	F
26	54	82		F	G	A	B	C	D	E
27	55	83		E	F	G	A	B	C	D
28	56	84		DC	ED	FE	GF	AG	BA	CB

The principle on which the last two Tables are constructed is easily explained.

In the first place, as seven Julian centuries contain exactly 25 Solar Cycles ($25 \times 28 = 7 \times 100$), it follows that, after the lapse of seven complete centuries, January 1 will fall on the same day of the week as it did the first of the seven centuries (1). For instance, in the century-column, calling the first century of the Christian Era 0, and knowing (as we do) that the first day of that century fell on Saturday, we have B for the Sunday Letter of A. D. 1. Therefore, January 1, A. D. 701, also fell on Saturday: so did January 1, 1401; and so on, through the successive cycles of seven centuries. Similarly, as January 1, A. D. 1 = January 1, B. C. 0, it results that January 1, B. C. 700, 1400, &c., also fell on Saturday, and so through the successive cycles of seven centuries B. C.

Again, as 100 Julian years contain 36,525 days, = 5217 weeks, *plus* 6 days, it follows that, in any two consecutive *centurial* years, the 1st of January in the one year will be 6 week-days apart from January 1 in the other (¹). Accordingly, as January 1, A. D. 1, A. D. 701, &c., fell on Saturday, and as January 1, B. C. 0, B. C. 700, &c., also fell on Saturday, we find—reckoning the 6 days *backwards* for the years B. C., and *forwards* for the years A. D.—the centurial week-days and Sunday Letters as follows:—

Jan. 1., B. C.	0 = Sat.,	B	Jan. 1, A. D.	1 = Sat.,	B
,,	100 = Sun.,	A	,,	101 = Fri.,	C
,,	200 = Mon.,	G	,,	201 = Thurs.,	D
,,	300 = Tues.,	F	,,	301 = Wed.,	E
,,	400 = Wed.,	E	,,	401 = Tues.,	F
,,	500 = Thurs.,	D	,,	501 = Mon.,	G
,,	600 = Fri.,	C	,,	601 = Sun.,	A
,,	700 = Sat.,	B	,,	701 = Sat.,	B
	&c.			&c.	

This being premised, the construction of the Tables is obvious.

The first four columns contain the successive years of each century, from 1 to 100. Each of the first three columns contains 28 numbers; and the number in each horizontal line differs by 28 from the one preceding it. The seven following columns contain, each, the Sunday Letters of the centuries at the head of these columns respectively. For instance, column 5 contains all the Sunday Letters of the first century of our era from A. D. 1, to A. D. 100. Column 6 contains the Sunday Letters of the second century, from A. D. 101, to A. D. 200, and so on. And since the cycles of seven centuries *recur* (as we have seen), the Tables exhibit all the possible cases; that is to say, the Sunday Letters for any number of centuries B. C., and A. D., in the *Julian* Calendar, or *Old Style*. The general expressions $\left(\frac{X}{7}\right)_r = 0$, $\left(\frac{X}{7}\right)_r = 1$, &c., show to which of the seven centurial columns any century *before* B. C. 4100, or *after* A. D. 4100, is to be referred. For instance, suppose B. C. 6900, or A. D. 6900: we have $\left(\frac{69}{7}\right)_r = 6$: therefore 69 belongs to the last column (11).

(1). This recurrence cannot take place sooner than after the lapse of 700 years, because $\frac{100}{28} = \frac{25}{7}$; and as 25 and 7 are prime to each other, their least common multiplier is 175; and $4 \times 175 = 700$.

(2). Or thus:—Let the Calendar Letters A, B, . . . G, be denoted by the numbers 1, 2, . . . 7. In 100 years there are 3 Solar Cycles, *plus* 16 years: and in these 16 years there are 20 regressions, $= 6$; for $\left(\frac{20}{7}\right)_r = 6$. Therefore, if n be the number of any Calendar Letter of any centurial year, the number of the same Calendar Letter the next centurial year will be $n - 6$, $= n + 1$, for $7 = 0$.

E. g.—Given that the *Sunday* Letter of A. D. 1401 is B (2), then the Sunday Letter of A. D. 1501 must be C ($= 2 + 1$).

31. A few examples will show the mode of using these Tables.

First, Table I., B. C.

Ex. 1. Required the Sunday Letter of B. C. 98. Here we have century 0 (head of column 5) and year 98 (like 14): these intersect in letter F, which is therefore the Sunday Letter sought.

Ex. 2. B. C. 100: A is the required Letter.

Ex. 3. B. C. 1720: 17 is found in column 8, and 20 in line 20; these intersect in Letter C, which is the Letter required.

Ex. 4. B. C. 6981: here $\left(\frac{69}{7}\right)_r = 6$; and therefore 69 is found in column 11: 81 is found in line 25, and these intersect in FG, which are the Letters required.

Secondly, Table II.

Required the Sunday Letters of the same years, A. D.

A. D. 98,	. . . . .	Sunday Letter	G.
A. D. 100,	. . . . .	,,	ED.
A. D. 1720,	. . . . .	,,	CB.
A. D. 6981,	. . . . .	,,	F.

32. The Julian Sunday Letter for any year B. C. or A. D. may also be found, without the use of Tables, by a simple arithmetical rule, which is thus investigated:—

In passing from one year to the next following year, the Sunday Letter *retrogrades* (Art. 24) one place, or two, according as the former year is a Common or Leap-year. Consequently, in passing from A. D. 1 to A. D. 2 there is *one* regression; from A. D. 2 to A. D. 3, another regression; and so on. Hence, if all years were Common ones, there would be $x - 1$ regressions in x years. But as each Leap-year causes two regressions, and as there are $\left(\frac{x}{4}\right)_w$ Bissextiles in x years, the additional number of regressions on account of the Bissextiles is the integer contained in $\left(\frac{x}{4}\right)_w$. Consequently, the *total* number of regressions in x years, is $x - 1 + \left(\frac{x}{4}\right)_w$. But this total number is composed of recurring sets, or circulating groups, of seven [1]. Hence, if we fix on any of the seven letters as the zero or starting point, and denote the other six letters, reckoned in *retrograde* order from it, by the numbers 1, 2, 3, 4, 5, 6, we may omit all the *complete* groups of seven, and take account of the remaining letters only. In other words, we may divide the total number of regressions by 7; and the remainder, applied to the scale just described, will show the number of places which the required Sunday Letter has receded from the Letter fixed upon as the zero or starting point. Now, to *find the starting point*, it is sufficient for us to know the Sunday Letter of A. D. 1. But as we know that Jan. 1, A. D. 1, fell on *Saturday*, the Sunday Letter of that year was B (Art. 27). Hence, taking B as the zero or starting point, and numbering the seven Calendar Letters in *retrograde* order, thus—

$$\begin{matrix} 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{A} & \mathrm{B}, \end{matrix} \qquad (A)$$

we have the *scale* whereby to ascertain the Sunday Letter of any year x, A. D. The following is the *General Rule for finding the Sunday Letter for any year* (x) A. D.:

To the year-number add its fourth part, omitting fractions; subtract 1 *from the sum: divide the result by* 7 *; and if there be no remainder, then B is the Sunday Letter: but if any number remain, then the Letter corresponding to that number in the annexed scale* (vid. *above*) *is the Sunday Letter.*

This Rule may be thus briefly formulated,— $L = \left(\frac{x + \left(\frac{x}{4}\right)_w - 1}{7}\right)_r$: where L is the number of the Sunday Letter; and $\left(\frac{x}{4}\right)_w$ denotes the integer *quotient* arising from dividing x by 4.

If the year x be a *Leap-year*, the Letter found by the scale is the *second* of the two Letters of the year; because in the above calculation *all* the Leap-years, and therefore all the regressions, are included.

One or two examples will suffice to illustrate this Rule:—

Ex. 1. Required the Sunday Letter of A. D. 325.

Here we have $x = 335$; $\left(\frac{x}{4}\right)_w = 81$; therefore, $\left(\frac{325 + 81 - 1}{7}\right)_r = 6$, which, applied to the scale, gives C for the required Letter.

Ex. 2. Required the Sunday Letters of A. D. 1500 (Leap-year).

Here we have $\left(\frac{1500 + 375 - 1}{7}\right)_r = 5 = \text{E}$, by the scale; and as E is the *second* Letter of the year, the required Letters are ED.

In the first Example the *quotient* of $\frac{405}{7}$ is 57; showing that 57 complete cycles of the seven Letters had elapsed. And in the second Example, the *quotient* of $\frac{1874}{7}$ is 267, showing that 267 complete cycles had elapsed.

It is easy to transform the numerical value of any Letter L, calculated in the above scale, into its numerical value calculated in the natural scale

$$\begin{array}{ccccccc} A & B & C & D & E & F & G \\ 1 & 2 & 3 & 4 & 5 & 6 & 7. \end{array} \qquad (B)$$

For, comparing the values of each Letter in both scales, we see that the sum = 9. Let, then, v be the value of any Letter in scale (A), u its value in scale (B); then

$$u + v = 9, \text{ or,}$$

$$u = \left(\frac{9 - v}{7}\right)_r = 7 + \left(\frac{2 - v}{7}\right)_r;$$

e. g. value of F in scale (A) is 3; ∴ in scale (B) it is 6.

Again, B = 0 in scale (A); ∴ in scale (B), $\text{B} = \left(\frac{9 - 0}{7}\right)_r = 2.$

Hence, substituting the value of v, viz. $L = \left(\frac{x + \left(\frac{x}{4}\right)_w - 1}{7}\right)_r$, we get

$$\text{value of } L \text{ in scale } (B) = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w - 3}{7}\right)_r$$

$$= 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w + 4}{7}\right)_r.$$

(1). A familiar illustration of this *circulating* process is furnished by a watch or clock. The zero, or starting point, on the dial is XII. In (say) a thousand hours after the hour-hand has passed that point, it will have arrived at IV., having made 83 complete revolutions, with four hours over. In short, $\left(\frac{1000}{12}\right)_r = 4$. So, in the case before us, the seven Calendar Letters correspond to the twelve hours of the dial plate; B to XII.; A to I.; and so on, in retrograde order, until B is reached again.

33. It is not necessary to make A. D. 1 (Letter B) the zero or starting point of the scale. Any other year A. D., with its corresponding Sunday Letter, will do as well. Our Prayer Book takes A as the zero, in the small annexed Table which it gives for the finding of the Dominical or Sunday Letter, and which is numbered as follows:—

$$\begin{array}{ccccccc} 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ B & C & D & E & F & G & A \end{array} \qquad (1)$$

In this scale we see that B, the Sunday Letter of A. D. 1, is marked 6; that is to say, the Sunday Letter of A. D. 1 is six places *behind* the zero point. The Letter of A. D. 2 is, therefore, seven places behind: that of A. D. 3, eight places; and, generally, supposing all the years *Common*, the Letter of A. D. x will have retrograded $x + 5$ places from A (the zero Letter). Adding $\left(\frac{x}{4}\right)_w$ regressions for the Bissextile years, we find the total number of regressions from zero A in x years, $x + \left(\frac{x}{4}\right)_w + 5$; but $5 = 7 - 2$; and, dividing by 7, 7 may be dropped. Consequently the Rule for finding the Julian Sunday Letter, with the *Prayer Book Scale*, is the following:—

Add to the year of our Lord its fourth part, omitting fractions; from the sum subtract the number 2: divide the result by 7; and if there be no remainder, then A is the Sunday Letter. But if any number remain, then the Letter corresponding to that number in the above scale is the Sunday Letter.

The Rule may, as before, be thus formulated:—

$$\left(\frac{x + \left(\frac{x}{4}\right)_w + 5}{7}\right)_r = \left(\frac{x + \left(\frac{x}{4}\right)_w - 2}{7}\right)_r;$$

because $5 = 7 - 2$, and 7 may be omitted, in the division by 7. This number to be referred to the above scale, in order to ascertain the corresponding Letter.

Ex. 1. Required the Sunday Letter of A. D. 325.

Here we have $\left(\frac{325 + 81 - 2}{7}\right)_r = \left(\frac{404}{7}\right)_r = 5 = C.$

Ex. 2. Required the Sunday Letters of A. D. 1500 (Leap-year). Here

$$\left(\frac{1500 + 375 - 2}{7}\right)_r = \left(\frac{1876}{7}\right)_r = 4 = D:$$

therefore the Letters are ED.

There is another form in which we frequently meet the scale presented—viz., by taking G as the zero, or starting point, thus

$$\begin{matrix} 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G}; \end{matrix} \qquad (2)$$

and it is easy to show, as above, that, with this scale,

$$L = \left(\frac{x + \left(\frac{x}{4}\right)_w + 4}{7}\right)_r = \left(\frac{x + \left(\frac{x}{4}\right)_w - 3}{7}\right)_r;$$

because $4 = 7 - 3$; and 7 may be omitted.

Ex. 1. A. D. 325 : $\left(\frac{325 + 81 - 3}{7}\right)_r = 4 = \mathrm{C}.$

Ex. 2. A. D. 1500 : $\left(\frac{1500 + 375 - 3}{7}\right)_r = 3 = \mathrm{D}.$

If the seven Calendar Letters be numbered consecutively in their *direct* order, thus,

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G}, \end{matrix} \qquad (3)$$

we see that the number of each Letter in (3) is the complement to 7 of the same Letter in (2); *e. g.*,

$$\mathrm{A}\ (3) = 7 - \mathrm{A}\ (2).$$

Hence we get for L the following formula, which will be found of much use:—

$$L = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w + 4}{7}\right)_r \qquad (4)$$

$$= 4 - \left(\frac{x + 1 + \left(\frac{x}{4}\right)_w}{7}\right)_r,$$

the scale being that of the natural order of the Calendar Letters from A . . . G. [1]

This Rule may be at once extended to the years *Before Christ*, by means of the *astronomical* mode of numbering the years B. C. (Art. 21): viz., calling the year next before A. D. 1 B. C. 0, instead of B. C. 1—thus

B. C. . . . . 4 3 2 1 ‖ 1 2 3 4, ... A. D.

In this way, since the year before A. D. was Leap-year, the Leap-years B. C. become multiples of 4, like those A. D.; and the same formula (4) will apply to both, by simply changing all the signs, as usual in passing through zero. Thus, calling L' the Sunday Letter corresponding to any year (x') B. C., we get

$$L' = \left(\frac{x' + \left(\frac{x'}{4}\right)_w - 3}{7}\right)_r - 7 = \left(\frac{x' + \left(\frac{x'}{4}\right)_w - 3}{7}\right)_r, \tag{5}$$

because 7 may be omitted.

N. B.—If the year be a Leap-year, the Letter so found is the first of the two.

In applying this formula it must be borne in mind that, as the astronomical date is one less than the chronological, $x' = x - 1$.

Ex. 1. Required the Sunday Letter of B. C. 165.

Here we have $x' = 164$ (a *Leap*-year); and, therefore, $L' = \left(\frac{202}{7}\right)_r = 6 = \text{F}$ by the scale. Thus, as the year is a Bissextile, and the formula gives the Letter for the part of the year *before* the intercalation, the required Letters are FE.

Ex. 2. B. C. 1502.

Here $x' = 1501$; and $L' = \left(\frac{1873}{7}\right)_r = 4 = \text{D}$.

(1). This is in effect Delambre's formula for finding L. His mode of investigating the problem is the following (*Astron. Mod.* i., p. 10):—The first year of the Christian Era began on Saturday; and, therefore, the Sunday Letter of it was B, = 2, the Calendar Letters being numbered in their natural order, from 1 to 7. A. D. 2 began with Sunday; and the Letter L, which was B (2) for A. D. 1, became A (1) for A. D. 2; that is to say, L became $L - 1$: and, in general, the Letter of *any* year being L, that of the next following year becomes $L - 1$; and after a number of years x, the Letter, becomes $L - x$. But as it will almost always happen that $x > L$, to make the subtraction possible we must add some multiple of $7 = n.\,7$. Thus (putting L' to express the number so formed) we get $L' = n.\,7 + L - x$. But when $x = 1$, $L = 2$, as before said; therefore, $L' = n.\,7 + 2 - x$, x being counted *after* the year A. D. 1. Now, since the Letter was 2 in the year A. D. 1, it was 3 in the year A. D. 0; consequently, reckoning x from A. D. 0, or (which is the same thing) making x = current year A. D., we get (omitting the trait over L', as unnecessary)—

$$L = n.\,7 + 3 - x.$$

We have not yet taken account of the Leap-years. Every Leap-year the number of the Letter diminishes by 2, instead of by 1: therefore the number of diminutions due to the Leap-years $= \left(\frac{x}{4}\right)_w$. Hence the value of L becomes

$$L = n.\,7 + 3 - x - \left(\frac{x}{4}\right)_w; \tag{1}$$

or, dividing the total number of diminutions into groups of seven, and making $n = 2$, we have

$$L = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w - 3}{7}\right)_r \tag{2}$$

$$= 7 - \left(\frac{x + 1 + \left(\frac{x}{4}\right)_w + 3}{7}\right)_r;$$

or, putting the quantity $x + \left(\frac{x}{4}\right)_w - 3 = R$,

$$L = 7 - \left(\frac{R}{7}\right)_r. \tag{3}$$

34. It is obvious that having found, in any of the above-mentioned ways, the *Sunday* Letter for any year, and, therefore, the days of each month on which Sunday falls, we get, at the same time, the days of each month corresponding to all the other days of the week, respectively, and *vice versâ*.

Ex. 1. Required the day of the week on which the 5th of April, A. D. 30, fell. We first find that the Sunday Letter for that year was A; which Letter, by the Table of Calendar Letters (Art. 23), belongs to April 2. Therefore the 5th of April fell on Wednesday.

Ex. 2. Required the days of the week on which January 1, and April 1, A. D. 1068, respectively fell. Here we find the Sunday Letters to be F E. F belongs to January 6; therefore January 1 fell on *Tuesday*. E belongs to June 1, therefore that day was Sunday.

Ex. 3. Required the day of the week on which April 5, B. C. 30, fell. Here we find that the Sunday Letter was E; which Letter belongs to April 6. Therefore the 5th of April fell on *Saturday*.

For the above Examples use was made of the Sunday Letter. But we can also solve the question *directly*, by finding, in the following way, the day of the week on which any given year begins.

We have seen already that each succeeding year begins one or two days *later* than its immediate predecessor, according as the latter is a Common or Leap-year. Hence the same formula will serve for the *advances* of the week-days as for the *regressions* of the Sunday Letters. We know that January 1, A. D. 1, fell on Saturday (Sunday Letter B): therefore A. D. 2 fell on Sunday, and so on. Accordingly, in the scale (A), Art. 32, we may write the week-days thus,

B	A	G	F	E	D	C
0	1	2	3	4	5	6
Sat.	Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.

And the total number of *advances* of the days, from A. D. 1 to A. D. x, like the number of regressions of the letters, will be $x + \left(\frac{x}{4}\right)_w - 1$; and, as before, the formula is

$$\left(\frac{x + \left(\frac{x}{4}\right)_w - 1}{7}\right)_r.$$

But it must be carefully noted that, if x be a Leap-year, the formula *includes* the intercalary day, and, consequently, in applying it to the 1st of *January*, we must diminish the total number of advances by 1.

Ex. 1. Required the week-day of January 1, A. D. 325.

Here we have $325 + 81 - 1 = 405$, and $\left(\frac{405}{7}\right)_r = 6 =$ Friday (by the scale).

The year 325 therefore began on Friday, as follows also from the Sunday Letter C.

Ex. 2. Required the week-day of January 1, A. D. 1500 (Leap-year).

We have $1500 + 375 - 2 = 1874 : \therefore \left(\frac{1873}{7}\right)_r = 4 =$ Wednesday (by scale).

Hence, knowing the day of the week on which the 1st of January falls any year, we can find at once, by the Table of Calendar Letters, the week-day corresponding to *any* day of any month [1].

Ex. 1. Find on what day of the week the 2nd of October fell, A. D. 325.

We have just found that the 1st of Jan. (Letter A) fell on Friday; therefore A will denote all the Fridays of that year. Now, the Calendar Letter of October 1 is A: therefore, October 2 was Saturday.

Ex. 2. Find the day of the week on which March 1st fell, A. D. 1500.

Jan. 1st (as we have just seen) fell that year on Wednesday; therefore A denoted all the *Wednesdays* up to Feb. 29; but the intercalary day will cause the following Wednesday to fall a day sooner than it would otherwise do, and consequently for the rest of the year the Calendar Letter of Wednesday will be G instead of A. But G belongs to March 4; hence, the 1st March fell that year on Sunday. Or it may be seen thus:—The 1st of March falls either three or four week-days further on than Jan. 1, according as the year is Common or Leap. Therefore, for the year 1500 (Leap-year), as Jan. 1 fell on Wednesday, March 1 fell on Sunday.

(1). Some writers (*vid.* Francœur, p. 276; and for the Gregorian Sunday Letter, p. 287) on the Calendar prefer to make March 1 the *characteristic* day of the year, instead of Jan. 1. One advantage of doing so is this—that, the intercalation, if any, being already made, the general expression requires no alteration for Leap-years. The *scale* only requires a slight alteration. Since in A. D. 1, Jan. 1 was Saturday, March 1 was Tuesday. Consequently, taking Tuesday as the starting point, we get the scale

B	A	G	F	E	D	C
0	1	2	3	4	5	6
Tues.	Wed.	Thur.	Fri.	Sat.	Sun.	Mon.

To this scale the same formula as in Art. 34 is to be applied.

Ex. Required the day of the week on which March 1 fell in A. D. 1500 (Leap-year).

$$\text{Here we have } 1500 + 375 - 1 = 1874, \therefore \left(\frac{1874}{7}\right)_r = 5.$$

By the scale Sunday is the required day. The Calendar Letter of March 1 is D, and therefore D denotes all the Sundays for the rest of the year.

35. It has now been shown how the *Sunday* Letter for any year A. D. or B. C. may be found in the Julian Calendar. This investigation does not exclusively concern the Ecclesiastical Calendar, but equally belongs to the Civil, and is intimately connected with some chronological problems. Accordingly, before passing on to the subject of *Easter*, which does specially concern the Ecclesiastical Calendar, it may be well to indicate a few of the more important cases in which the results arrived at may be applied to the calculation of intervals of time. I have incidentally noticed some questions of this nature already; but it will be useful to dwell a little more on the subject. It will be borne in mind that for the present I confine myself entirely to the *Julian Calendar before the Gregorian correction*; in other words, to *Old Style*.

In determining the interval of time between any two assigned dates, it is important to attend to the following points:—

1. A *date*, whether it be a day or a year, denotes the day or year *current*, not *elapsed*: *e.g.*, March 22 means that 21 days had elapsed, and that the 22nd was current; A. D. 10 expresses that 9 years had elapsed, and that the 10th was current.

2. In calculating the *interval* defined by two given dates, the earlier day or year is usually reckoned *inclusively*; the later *exclusively*; in other words, "*from*" is inclusive, "*to*" is exclusive: *e.g.*, the number of days *from* Jan. 1 *to* Feb. 1 is 31; the number of years *from* 1801 *to* 1874 is 73.

But the opposite may be *implied* or *expressed*; that is to say, the earlier day or year may be *exclusive*, the later one *inclusive*. When *neither* extreme is included, the interval is, of course, one less.

3. The *Civil* day is reckoned as beginning at midnight, and ending the following midnight. The *Astronomical* day begins at noon, and ends on the following noon; and is therefore twelve hours behind the Civil day.

36. The following are specimens of the kind of questions referred to in the last Article.

(*a*). *How to find the number of days elapsed from any given day of the year to any other given day of the same year.*

This question has been already answered, Art. 11, and Art. 23.

(*b*). *To find the number of days elapsed from the first of Jan.*, A. D. 1, *to the* 1*st of Jan.*, A. D. x.

If all the years were *Common*, the number of days would be $(x - 1). 365$. The additional number, due to the Leap-year, is $\left(\frac{x-1}{4}\right)_w$.

Therefore, the total number of days (D) is

$$D = (x - 1). 365 + \left(\frac{x-1}{4}\right)_w. \quad (1)$$

Ex. Required the number of days *from* Jan. 1, A. D. 1, *to* Jan. 1, A. D. 1401.

$$\text{Here we have } 1400 \times 365 + \left(\frac{1400}{4}\right)_w = 511, 350.$$

(*c*). *To find the number of days from any given day*, A. D. 1, *to any given day*, A. D. x.

First, find (by (*b*)) the number of days *from* the given date, A. D. 1, *to* the *same* date, A. D. x; and then find (as in (*a*)) the number of days from the latter to the given date that year.

Ex. 1. Find the number of days *from* April 10, A. D. 1, *to* August 4, A. D. 441.

First, we have from April 10, A. D. 1, to April 10, A. D. 441 (= Jan. 1, A. D. 1, to Jan. 1, 441), the number of days $= 440 \times 365 + \left(\frac{440}{4}\right)_w = 160, 710$;

and (by Table II., Art. 11, or Table Art. 23) from April 10 to Aug. 4 there are 116 days (216 − 100).

Hence, the number of days required = 160,826.

Ex. 2. Required the number of days from April 10, A. D. 1, to August 4, A. D. 444.

Here we must add 1 on account of the *Leap*-year, because the general formula in (*b*) extends only *to* Jan. 1, 444. Hence, from April 10, A. D. 1, to April 10, A. D. 444 (= from Jan. 1, A. D. 1, to Jan. 1, A. D. 444, *plus* 1) we get

$$443 + \left(\frac{443}{4}\right)_w + 1 = 161,806$$

and from April 10 to Aug. 4, = 116

Total, . . 161,922 days.

(*d*). *Required the number of days from Jan.* 1, A. D. x, *to Jan.* 1, A. D. x'. By the formula (1) we have

$$D = (x - 1).\ 365 + \left(\frac{x-1}{4}\right)_w;$$

and
$$D' = (x' - 1).\ 365 + \left(\frac{x'-1}{4}\right)_w.$$

Therefore, subtracting D from D', we get

$$D' - D = (x' - x) \times 365 + \left(\frac{x'-x}{4}\right)_w. \qquad (2)$$

But in certain cases, viz., when $\left(\frac{x'-1}{4}\right)_r < \left(\frac{x-1}{4}\right)_r$, we must add 1 to this result (Art. 87, note 2).

Ex. 1. Required the number of days from Jan. 1, 100, to Jan. 1, 1500.

Here we have $x' - x = 1400$; and $1400 \times 365 = 511{,}000$

$$\left(\frac{x'-x}{4}\right)_w = \left(\frac{1400}{4}\right)_w \qquad = \quad 350$$

511,350 days.

Ex. 2. No. of days from Jan. 1, 100, to Jan. 1, 1501.

Here the correction of 1 comes in, and interval = 511,716.

(*e*). *Find the number of days from any given day*, A. D. x, *to any other given day*, A. D. x'.

Let a be the *number* of the former day, and a' that of the latter. Then D becomes $D + a$; and D' becomes $D' + a'$: and formula (2) becomes

$$D' - D = (x' - x) \times 365 + \left(\frac{x'-x}{4}\right)_w + a' - a\ [+ 1]. \qquad (3)$$

The number 1 within the last bracket is only to be added when $\left(\frac{x'-1}{4}\right)_r < \left(\frac{x-1}{4}\right)_r$.

There are three cases to be considered.

(α). $a' = a$: in this case, the number of days is given by formula (2).

Ex. The number of days from Jan. 10, A. D. 100, to Jan. 10, A. D. 1500, or as above, = 511,350.

(β). $a' > a$.

Ex. Required the number of days from Jan. 6, A. D. 1001, to Sept. 26, A. D. 1500.

From Jan. 6 to Jan. 6 (= Jan. 1 to Jan. 1), ... 511,350 days.

From Jan. 1 to Sept. 26 (Leap-year), by Table Art. 23, + 264

$a' - a = 264$. Total, . . 511,614 days.

(γ). $a' < a$.

Ex. Required the number of days from Nov. 1, A. D. 400, to March 25, A. D. 1204.

Here we have $x' - x = 804$.

$$\text{Hence, } 804 \times 365 + \left(\frac{804}{4}\right)_w = 293{,}661 \text{ days.}$$

$$\begin{aligned} a' &= 85 \text{ (Leap-year).} \\ a &= 306 \text{ (,,).} \end{aligned}$$

$$a' - a = -221 \quad . \quad . \quad . \quad . \qquad -221$$

The number of days sought = 293,440

(δ). Find the number of days from Nov. 1, A. D. 400, to March 25, A. D. 1205.

Here $\left(\frac{x'-1}{4}\right)_r < \left(\frac{x-1}{4}\right)_r$, and we therefore add unity to the result we should otherwise obtain. So interval $= 805{,}365 + \left(\frac{805}{4}\right)_w + 1 + 84 - 306 = 293{,}805$ days.

37. Let us next consider the years *before Christ.*

Let x be the *Chronologer's* number (Art. 21) of any year B. C. It is easy to show that the number of Leap-years corresponding to x is $\left(\frac{x+3}{4}\right)_w$. For as B. C. 1 is a Leap-year, and each preceding group of four years gives a Leap-year, it follows that the total number of Leap-years $= 1 + \left(\frac{x-1}{4}\right)_w = \left(\frac{x+3}{4}\right)_w$.

Now, supposing all the years B. C. to be *Common*, the number of days in x years would be $x.365$: the number of Leap-years we have just seen to be $= \left(\frac{x+3}{4}\right)_w$; therefore the total number of days *from* B. C. x *to* Jan. 1, A. D. 1, is

$$x \times 365 + \left(\frac{x+3}{4}\right)_w. \qquad (1)$$

Ex. Required the number of days *from* Jan. 1, B. C. 501, *to* the first day of the Christian Era.

$$\text{Here we have } 501 \times 365 + \left(\frac{504}{4}\right)_w = 182{,}991 \text{ days.}$$

Hence, we can find the number of days from Jan. 1, B. C. x', to Jan. 1, B. C. x (x' being the greater).

As in Art. 36, we have

$$D' - D = (x' - x)\ 365 + \left(\frac{x' - x}{4}\right)_w [+ 1], \qquad (2)$$

the number 1 within brackets to be added only when $\left(\frac{x' - 3}{4}\right)_r < \left(\frac{x - 3}{4}\right)_r$ (Art. 87, Note 2).

Ex. Find the number of days from Jan. 1, B. C. 100, to Jan. 1, B. C. 8.

Here we have $x' - x = 92$, and therefore

$$D' - D = 92 \times 365 + 23 = 33{,}603 \text{ days}.$$

To find the number of days from any given day B. C. x' to any given day B. C. x. Let a' and a be the numbers of the given days, respectively: then we have, as in Art. 36 (3),

$$D' - D = (x' - x)\ 365 + \left(\frac{x' - x}{4}\right)_w + a' - a\ [+ 1], \qquad (3)$$

the additional unit to be included only when $\left(\frac{x' - 3}{4}\right)_r < \left(\frac{x - 3}{4}\right)_r$.

38. We can now find the number of days between Jan. 1, B. C. x, and Jan. 1, A. D. y.

The number of days *from* Jan. 1, B. C. x, *to* Jan. 1, A. D. 1, is (Art. 37 (1))

$$D = x.\ 365 + \left(\frac{x + 3}{4}\right)_w;$$

and the number of days *from* Jan. 1, A. D. 1, *to* Jan. 1, A. D. y, is (Art. 36 (1))

$$D' = (y - 1)\ 365 + \left(\frac{y - 1}{4}\right)_w.$$

Hence, the required number of days is

$$D + D' = (x + y - 1).\ 365 + \left(\frac{x + 3}{4}\right)_w + \left(\frac{y - 1}{4}\right)_w. \qquad (1)$$

Consequently, we get this rule:—*From the sum of the years* B. C. *and* A. D. *deduct* 1; *multiply the result by* 365; *find the quotients arising, respectively, from dividing by* 4 *the numeral of the year* B. C. *increased by* 3, *and the numeral of the year* A. D. *diminished by* 1; *add these quotients to the result previously obtained.*

Ex. Required the number of days from Jan. 1, B. C. 1586, to Jan. 1, A. D. 70.

Here we have $(1586 + 70 - 1).\ 365 + \left(\frac{1589}{4}\right)_w + \left(\frac{69}{4}\right)_w = 604{,}489$ days.

Similarly, we can find the number of days *from any* date (a) B. C. x, *to* any date (a'), A. D. y. Here D becomes $D - a$; and D' becomes $D' + a'$: therefore,

$$D + D' = (x + y - 1)\ 365 + \left(\frac{x + 3}{4}\right)_w + \left(\frac{y - 1}{4}\right)_w + a' - a. \qquad (2)$$

Ex. 1. Required the number of days *from* April 10, B.C. 1586, *to* Aug. 4, A.D. 70.

We have, as already found, from Jan. 1 to Jan. 1, 604,489 days.

$$\left.\begin{array}{l} a' = \text{August 4}, \quad . \quad . \quad . \quad 216 \text{ days} \\ a = \text{April 10}, \quad . \quad . \quad . \quad 100 \quad ,, \end{array}\right\} \therefore a' - a \; = 116$$

Total, . . . 604,605 days.

Ex. 2. Find the number of days between the epoch of the Era of the Seleucidæ (Oct. 1, B.C. 312) and the epoch of the Era of the Hejirah (July 15, A.D. 622).

Here we have $(x + y - 1) \times 365 = 933 \times 365 = 340{,}545$ days.

$$\left(\frac{315}{4}\right)_w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad = \quad 78$$

$$\left(\frac{621}{4}\right)_w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad = \quad 155$$

340,778 days.

Both the given years are Common : therefore,

(a) October 1 = 274 days.

(a') July 15 = 196 ,,

Therefore $a' - a$ . . . = -78

Total number of days sought = 340,700 days,

agreeing with what we have already found, Art. 21, Note 2.

39. The calculation of intervals of time may also be easily effected by the following method, based on the Julian *Quadriennium*, or period of four years, which contains 1491 days (365 × 3 + 366).

The method is as follows :—Divide the number of years in the given interval by 4. Let Q be the quotient, and R the remainder. Then Q will show the number of complete quadriennia of 1461 days each, and R will be the residual years, which cannot exceed *three*.

These residual years may be *all* Common, but only *one* can be a Leap-year. The total number of days in $Q + R$ will be the number of days from Jan. 1 of the first given year to Jan. 1 of the second given year.

Before showing by examples the application of this method, it will be well to premise the following Tables.

1	Quadriennium = 1,461	days.	7	Quadriennium = 10,227	days.
2	,, = 2,922	,,	8	,, = 11,688	,,
3	,, = 4,883	,,	9	,, = 13,149	,,
4	,, = 5,844	,,	10	,, = 14,610	,,
5	,, = 7,305	,,	11	,, = 16,071	,,
6	,, = 8,766	,,	12	,, = 17,532	,,
2	Common years = 730	,,	1	Com. 1 Biss. = 731	,,
3	,, ,, = 1,095	,,	2	Com. 1 Biss. = 1,096	,,

Ex. 1. Required the number of days *from* July 9, B. C. 588, *to* Nov. 17, A. D. 70. *From* July 9, B. C. 588, *to* July 9, A. D. 70, there are 657 years = 164 quadriennia *plus* 1 Common year (A. D. 69).

164 quadr. =	239,604 days.
1 year =	365
	239,969
From July 9 to Nov. 17 =	131
Total, . .	240,100 days.

Ex. 2. Required the number of days *from* April 10, B. C. 1586, *to* Aug. 4, A. D. 70.

Here we have from April 10 to April 10, an interval of 1655 years, = 413 quadr., *plus* 3 years, of which one (A. D. 68) is Bissextile.

Hence, we have

413 quadr. =	603,393 days.
3 years (1 Biss.) =	1,096
April 10 to Aug. 4 =	116
Total, . .	604,605 days,

as found before (Art. 38).

40. I now proceed to show how *Easter* Sunday, in particular, as the centre of the group of Moveable Feasts depending on it, is found in any given year. I shall not here stop to discuss the old and long-debated questions relative to what is known as the "Easter Controversy"; nor shall I examine in detail the various steps by which the Christian Church finally arrived at a universal agreement respecting the mode in which Easter Day should be determined. These subordinate questions I have dealt with in an Appen-

dix. At present I shall content myself with merely glancing at such points as are necessary for the clear understanding of the subject immediately before us.

The determination of Easter involves, by the very terms of its definition, the consideration of the Lunar, as well as that of the Solar, year. The Lunar year consists of 12 Lunar *synodic months;* each month being the interval between two consecutive conjunctions or oppositions of the Sun and Moon; in other words, between one New Moon and the next, or one Full Moon and the next. The synodic month is not of invariable length. Its *mean* length, with which alone we are concerned (1), is $29^d\ 12^h\ 44^m\ 2{\cdot}87^s$, or, reduced to decimals, 29·530588715 days. This is called a mean *Lunation.* Before its exact length was ascertained, it was reckoned as $29\frac{1}{2}$ days (nearly $44^m\ 3^s$ too little). But as fractional measures of time are unsuited to the ordinary purposes of life, it was usual among the nations who made use of the Moon as a measure of time to reckon the Lunar month as containing 29 and 30 days *alternately.* But neither the approximate mean month of $29\frac{1}{2}$ days, nor the true mean month, is an exact measure of the mean Solar, or Tropical, year. Twelve Lunar months, of 29 and 30 days, alternately, amount to 354 days; which fall short of a Common Julian year by 11 days. And 12 true mean Lunations amount to 354·367064 days; falling short of a mean Tropical year (365·242216) by 10·875152 days. This incommensurability between the Lunar and the Solar years has rendered it necessary to adjust them, as nearly as possible, by means of *cycles*, as shall be presently shown.

(1). It must be borne in mind, generally, that in all the Church's reckoning respecting Moveable Feasts, it is not the *real* motions of the Sun that are made use of, but the *mean* motions; and, in the case of the Moon, certain *cycles* have been always employed, as we shall see further on. The real motions of these luminaries, being variable, and also difficult to ascertain with perfect accuracy, have not been employed by the Church for the fixing of her Festivals. Even if *perfect* astronomical tables could be constructed, still, on account of the difference of meridians, the exact determination of the true motions of the Sun and Moon would be totally useless for ensuring the uniform observance of Easter Day. In fact, unless some one " Easter Meridian " were agreed on by the whole Christian world, the simultaneous observance of an *astronomically* exact Easter would be impossible. Suppose, for example, that in the meridian of London, an *astronomical* Full Moon were to fall very early in the morning of Saturday, March 21, then the next day would be Easter Day for all places on that meridian. But that same Full Moon would, in places sufficiently to the west of London, happen on Friday night, March 20; and, therefore, in those places it would not be the Paschal Moon at all, since the Paschal Full Moon must fall on or after March 21. Clavius, who clearly saw all this (*vid.* Calend. Roman. pp. 77, 389, 566), accordingly emphatically declares that the Church is not tied, and never was, to the *true* motions of the Sun and Moon; and that it is sufficient for her purpose if she can approximate, *pingui quâdam Minervâ*, to the motions of those bodies, provided she does not recede much from their true motions.—*Vid.* more on this subject below, Arts. 44 and 150.

41. The Conditions for determining Easter Day, as laid down in the old Church Calendar, and which still remain unchanged, are the four following (1): —

I. *It must be kept on Sunday.*

This Rule was in direct opposition to the Quartodecimans, who kept Easter on the third day after the 14th of the Jewish month Nisan (inclusive), whatever day of the week it might fall upon. This Rule was established in the Roman Church as early, at least, as the middle of the second century (2).

II. (*a*) *This Sunday must be the next after the* 14*th day of the Paschal Moon, reckoned from the day of the New Moon* (3) *inclusive.*

(*b*) *If the* 14*th day should happen to be Sunday, then Easter must not be kept until the following Sunday.*

This Rule also was established in the early Roman Church. Pope Victor, at the close of the second century, states it as a fundamental principle in the observance of Easter (4).

III. *The Paschal Moon is that Calendar Moon whose* 14*th day falls on, or next follows, the day of the Vernal Equinox.*

The month in which this takes place was called (the Jewish phraseology being retained) the *First Month.*

IV. *The* 21*st of March is to be taken as the invariable day of the Vernal Equinox* (5).

The above conditions are briefly summed up in The Definition of Easter given in the Roman Breviary:—"Ex decreto sacri Concilii Nicæni, Pascha, ex quo reliqua Festa Mobilia pendent, celebrari debet die Dominico qui proxime succedit xiv. Lunæ primi mensis; is verò apud Hebræos vocatur primus mensis cujus xiv. Lunæ vel cadit in diem Verni Equinoctii, quod die 21 mensis Martii contingit, vel propius ipsum sequitur." (6)

(1). The authority of the General Council of Nicæa is claimed for these Rules. But *vid.* Appendix.

(2). *Vid.* Appendix.

(3). *Vid.* Appendix.

(4). This Rule was regarded as essential, in order to prevent Easter from being kept either *before* the day of the Jewish Passover (which would be absurd, as putting the Resurrection Day before the Passion Day), or *on* the Passover Day—a coincidence which the prejudices of the Christians against the Jews would not tolerate. But it must be remembered that the New Moon intended by Pope Victor, and by the Nicene Fathers, was most probably the *real* New Moon; or, if it were the New Moon of a *cycle*, it was not that of the *nineteen-year* cycle, which was not finally adopted by the Church for the regulation of Easter until long after.

(5). It was deemed essential that Easter should be celebrated either at the actual Full Moon, or after it, but *never before*, lest the celebration should take place, not in the *first* month, but in the *last* of the preceding year.—*Vid.* also Note 3 to Art. 80. It is curious to find the Christian Church holding so fast to the Jewish *first month*, while, at the same time, it took such pains that Easter should not fall on the same day as the Jewish Passover. As the Jews kept their Passover, and the Quartodecimans their Easter, on the 14th day of the real Moon—that is, the day before the actual Full Moon—the great object of the Easter Rules was to prevent Easter being kept either *before* the Jewish Passover (which would be absurd), or *on* the Passover Day. Provided this point were gained, there was no objection to Easter being kept *on the day* of the actual Full Moon; on the contrary, the *sooner* it was kept *after* the Passover Day, the better.—*Vid.* De Morgan, p. 18. Neither could Easter be legitimately celebrated in the *second* month, *i. e.* the month whose real Full Moon was the *second* after the Equinox. Nevertheless, both these irregularities occurred frequently, though unavoidably, under the Old Calendar, when the Paschal Moons were regulated by the Golden Numbers. In the Reformed Gregorian Calendar, in which Epacts are substituted for the Golden Numbers of the Old Calendar, the irregularity in question, though not rendered absolutely impossible, can very rarely occur.

(6). This was the date calculated or assumed for the Vernal Equinox in the year 325 A. D., by those who finally settled the Easter question. According to Delambre's Solar Tables, the following are the respective dates of the Vernal Equinox for the three important Chronological Eras—the Introduction of the Julian Calendar, the Nicene Council, and the Gregorian Reformation:—

45 B. C., ...	March 23, about 5 A. M.	The hours are for the Meridian of Berlin and Rome.
325 A. D., ...	,, 20, ,, 2 P. M.	
1582 A. D., ...	,, 11, ,, 1 A. M.	

42. It must be carefully borne in mind that the *Equinox* intended in these Easter Canons is not the *true* Equinox, but what may be called, in contradistinction to it, the *Ecclesiastical* or *Calendar* Equinox (*vid.* Clavius, cap. v. § 13). The true, or *actual* Equinox, cannot be fixed to a *single* day (March 21), because, in consequence of the Bissextile year it necessarily vibrates between *two* days. It is easy to see the reason of this. For, assuming (in accordance with the Copernical Solar Tables) that, in the year 324 A. D., the true Equinox fell on March 21, about noon; then, as the Common year of 365 days is nearly 6 hours too short, the Equinox would fall the next year (325) about 6 P. M.; the following year (326) about midnight; the third year (327) about 6 A. M., March 22; and the fourth year (328) about noon again. But this fourth year being again *Leap*-year, the effect of the intercalary day in February was to bring everything after the intercalation a nominal day *earlier;* so that the actual Equinox in the year 328 was again nearly restored to the noon of March 21; and the same process of alternation from March 21 to 22 would be repeated the next four years, and so on continuously, if the length of the Julian year had been rigidly exact.

Hence it is evident that the "Vernal Equinox" in Rule III. must mean the *Calendar* Equinox, *not* the *actual.* Otherwise, whenever, on a third year after Bissextile, the 14th day of a Calendar Moon happened to fall on March 21 (which it always did when the

Golden Number of the year was XVI.), that fourteenth Moon would not be a Paschal Moon, as happening *before* the true Equinox (March 22); and would not become so until the following month. With respect to the *true* Equinox, the fourteenth Moon in question would belong, not to the *first* month, but the *last* of the preceding year. But the Church always regarded it as the Paschal Moon, inasmuch as it fell on March 21, and Easter was accordingly celebrated on that last month, and not on the following, which, in reference to the *true* Equinox, was the *first* month of the new year. Hence it is evident the Equinox intended in Rule III. was the *Calendar* Equinox, fixed to a *single* day (March 21), and not the true Equinox, which necessarily vibrates between two days. The ancient computists were, no doubt, well aware of this vibration; and that, in fixing the Calendar Equinox to March 21, Easter must, in the case referred to, be celebrated in the *last* month. But the main object of the Church of the Nicene age being that Easter should be celebrated, all over the Christian world, on one and the same day of the year, it was resolved to secure this object by fixing the Ecclesiastical Equinox to a *single* day (March 21), even though this day might *precede* or *follow* the true Equinox by a short interval, and so, in reference to the *true* Equinox, Easter might be celebrated in the *last* or *second* month of the year, instead of the *first*.

43. In what has just been said we have assumed that the true Equinox, at the date of the Nicene Council, fell on March 21; and we have seen that the *fixing* of the Equinox (Rule IV.) to this date necessarily gave occasion to the celebration of Easter, in a certain contingency, in the *last* month instead of the *first*. We have next to consider what the result was, supposing that the *true* Equinox fell (as according to Delambre's Solar Tables it did actually fall) A. D. 324, on March 20, at 8 A. M. The following year (325) it fell about 2 P. M.; the next year (326), about 8 P. M.; the third year (327), about 2 A. M., March 21; and the fourth year (328), about noon again. But, as already explained, the effect of the Bissextile day was to restore the Equinox, this fourth year, to the place which it occupied four years before—viz., 8 A. M., March 20. This vibration between March 20 and 21 would go on regularly, if the Julian year were exact. In this case, the effect of *fixing* the *Ecclesiastical* Equinox to March 21 necessarily would be that, whenever a fourteenth day of the Moon, in a Bissextile year, or either of the two years after it, fell on March 20, Easter was kept in the *second* month; because, although that fourteenth Moon was a true Paschal Moon in reference to the *actual* Equinox (March 20), yet the Church did not consider it such, inasmuch as it *preceded* the Ecclesiastical Equinox (March 21); and, therefore, Easter was not kept until the next fourteenth Moon, that is to say, until the *second* month of the year in relation to the *true* Equinox. In the Old Calendar, care was taken that no fourteenth Moon should ever fall

on March 20, by not affixing any Golden Number to March 7. But in the Gregorian Calendar the case in question arises whenever the *Epact* of the year is 24; because this Epact belongs to March 7, and the fourteenth day, reckoned from it inclusively, is March 20. But when this occurs, the Church's practice, in accordance with Rules III. and IV., is to keep Easter the following month.

44. The general result of all this is, that the Church's decrees respecting the dependence of Easter on the Vernal Equinox are to be understood of the *Ecclesiastical* Equinox, fixed to March 21, and not of the true Equinox, which vibrates between two days: the former is to be always observed, even though the true Equinox may precede or follow it, provision being made so that the divergence between them shall be small (1).

It is obvious also (as De Morgan observes, p. 13) that this source of occasional inaccuracy in the determination of Easter, arising from the vibration of the Equinox, would still exist even if the Moon of the Calendar were the actual Moon of the heavens, which it is not. For, if a fourteenth day of the actual Moon were to fall on March 21, at a time when the true Equinox falls on March 22, that fourteenth would be held to be the Paschal fourteenth, and Easter Day would be the following Sunday, even though the fourteenth Moon fell before the Equinox. Hence a difference of a whole Lunar month would be occasioned in the time of celebrating Easter.

(1). The following passage of Clavius (cap. v. §§ 12, 13) is worthy of quoting in connexion with the subject of the above section:—"Ex his liquido apparet, novum non esse in Ecclesia, ut Pascha vel in ultimo mense, vel in secundo celebretur, propter Æquinoctium uni diei affixum, quod duos dies postulat. Neque vero in hoc Ecclesia a Decretis Patrum et Concilii Nicæni discedere putanda est: quia Decreta illa non ita severe accipienda sunt ut velint Ecclesiam ex tabulis astronomicis Æquinoctium debere explorare; sed ita solum sunt intelligenda ut Ecclesiam præcipiant in Paschæ celebratione observared ebere diem Æquinoctio ascriptum, licet nonnunquam Æquinoctium antecedat illum diem, vel subsequatur, dummodo non longe ab eo recedat. Quocirca rectè dici potest Æquinoctium duobus modis accipi posse; *uno*, prout vere in rerum natura existit, et ab astronomis consideratur; *alio*, prout in Calendario affigitur ad certum aliquem diem, et ab Ecclesia ad Pascha rite celebrandum adhibetur. Illud astronomicum, sive verum; hoc ecclesiasticum, sive politicum, dicatur. Et quamvis ecclesiasticum hoc Æquinoctium debeat ad astronomicum illud ac verum quantum fieri potest referri, justas tamen ob causas potest unum ab altero discrepare. Justissima autem causa quare ecclesiasticum Æquinoctium a naturali ac vero interdum differat est concordia fidelium atque consensus, propter quem Ecclesia Æquinoctium unico diei ascripsit, ut nimirum in orbe Christiano universo eodem die conjunctis animis Sacrosanctum Paschæ diem omnes agitarent; quæ sane concordia vix, et ne vix quidem, retineri posset, si Æquinoctium, habitâ ratione veri motus, ad duos aut etiam plures dies alligaretur. Itaque si casu aliquo detrimentum accipiendum sit vel in Æquinoctio motuque Lunæ, vel in concordia fidelium, exoptet magis, Pascha eodem die ab omnibus celebrari, etiamsi non perfectè Æquinoctium cursusque Lunæ teneatur, quam vel minimam consensionem fidelium in eo celebrando dirimi aut perturbari."

45. It must also be carefully observed (as above intimated) that the Moon, whose 14th day is spoken of in the above Rules, is *not the real Moon; nor* yet *the mean Moon* of astronomers, but the *fictitious Moon* of the Lunar Cycle, to which the name "Ecclesiastical," or "Calendar," has been given (1). The age of this Calendar Moon does not usually coincide with that of the real, or even the mean, Moon; but the difference does not exceed a certain limit. The Calendar Full Moon may differ as much as two, or even three days from the real Full Moon, but not more. Clavius expressly says (cap. xviii., § 4) that a Lunar Cycle must be selected such that the 14th of the Calendar Moon given by it shall not *precede* the mean Full Moon by more than one day, or fall more than two days *after* it. There is no objection whatever against the Calendar Full Moon falling *on* the same day as the real or mean Full Moon.

(1). *Vide* Art. 40, Note.

46. The conditions for determining Easter involve, as already observed (Art. 40), both the Solar and Lunar years. That Easter Day must be a Sunday, and one subsequent to the Vernal Equinox, involves the consideration of the Solar year; while the relation which that Sunday must bear to the Calendar Moon's age introduces the Lunar month and year as a necessary element in the calculation. Now, the Solar year and the Lunar synodic month, or Lunation, are (Art. 40) incommensurable quantities; and so, of course, are also the Solar year, and the Lunar year, which contains twelve Lunations. For facility of calculation it became necessary to find some convenient *approximate* common measure of those incommensurable quantities, the Solar and Lunar year. The problem to be solved was to find an integer number of Solar years, such that, supposing the Sun and Moon to be in conjunction (New Moon), or in opposition (Full Moon), at any epoch, they shall be *q. p.* in conjunction or opposition again, after the lapse of that number of years. Several such intervals of time, or *cycles*, as they were called, were discovered in ancient times, and were made use of for the determination of Easter (1). But the most famous of all, combining at once great exactness with practical convenience, was the nineteen-year cycle (ἐννεακαιδεκαετηρὶς), discovered by Meton, a celebrated Athenian astronomer, about the year B. C. 433, and which was called from him the *Metonic Cycle*. Some have thought that the discovery of this cycle is not due to Meton, but that he merely learnt it in Egypt, whither most of the early Greek philosophers repaired to study the sciences, especially astronomy (2). However this may be, the cycle was publicly adopted at Athens in the year 433. It embodied the beautiful discovery that in 19 Solar years (of 365¼ days, each) there are almost exactly 235 actual Lunations; so that after the completion of every successive cycle of 19 years, the New Moons (and, therefore, all the other phases) recur in the same order, on the same days

of the month, and very nearly at the same hours of the day, as they did 19 years before. Thus, if 1, 2, 3, . . . 19 denote the successive years of any cycle, and 1′, 2′, 3′ . . . 19′ the corresponding years of the next (or, of any preceding or succeeding) cycle, then the New Moons of year 1′ will fall on the same month-days, and very nearly at the same hour, as those of 1; and the same will hold good of 2′ and 2, 3′ and 3, and so on. It is obvious that if a Table be constructed showing the month-days on which the successive New Moons fall during any cycle of 19 years, that Table will serve, without any further trouble, to determine the New Moons for all preceding and succeeding cycles. And the same, of course, is true of the Full Moon. To find the New (or Full) Moons in any given year of any cycle, it would be sufficient merely to look at those of the corresponding year in the Table. This great discovery excited such admiration and enthusiasm at Athens, that Meton himself was honoured with an Olympic crown (3); and the successive years of the cycle, with the dates of the Full Moons corresponding to each year, in the Table or Canon constructed by him, were ordered to be inscribed in characters of gold upon the walls of the Temple of Minerva, and (it is said) on other public monuments also; and when this cycle was finally adopted, with certain modifications, by the Christian Church, for the determination of Easter, the old practice was continued, of writing the 19 consecutive years of the cycle in gilt characters, in the Mediæval Church Calendars. Hence the origin of the name Golden Numbers (4).

(1). Such cycles were the ὀκταετηρίς, or eight-year cycle, of Cleostratus (*vid. Ideler*, i., 294; ii., 605); the δωδεκαετηρίς, or twelve-year cycle, of the Chaldeans (*Ideler*, i., 301); the ἑκκαιδεκαετηρίς, or sixteen-year cycle, of Hippolytus (*Ideler*, ii., 213); the ἑκκαιεβδομηκονταετηρίς, or seventy-six-year period, of Callippus (*Ideler*, i., 299); the eighty-four-year cycle of the ancient Jews (*Ideler*, i., 571); the ninety-five-year cycle of Cyrill (*Ideler*, ii., 258).

(2). Laplace (Systeme du Monde, p. 365) claims the discovery of this cycle for the Chinese, sixteen centuries before Meton's time! and Bailly also (Hist. de l'Astron. Anc.) says that the cycle was known to them and the Indians long prior to Meton's age. The modern Chinese use this cycle; but there is no evidence for the assertion that they knew it 2000 years before Christ. Ideler (i., 313; ii., 608) has successfully vindicated the first discovery of the cycle to the Athenian philosopher (Handb. vol. i., 313; vol. ii., 608).

(3). It is a curious fact that Aristophanes ridiculed him (in the Comedy of the "Birds," *vv.* 991, *sqq.*) as a person whose head was filled with all kinds of fantastic and useless speculations.

(4.) This expression originated in the Middle Ages, and probably after Bede's time, as he makes no mention of it. Vid. *Ideler*, ii., 197, N.

47. The Lunar synodic month was known to consist of 29½ days, *plus* a small fraction of a day. Accordingly, in order to apply Meton's discovery to the construction of such a Table as that just described, it was necessary to substitute for the 235 *actual*

Lunations the *same number* of *Calendar* months, each consisting of an *integer* number of days, and varying but little from the actual length of a Lunation; and, moreover, such that the whole 235 would, as nearly as possible, be equal to $6939\frac{3}{4}$ days. There is some diversity of opinion among modern writers on Chronology respecting the exact method adopted by Meton in the construction of his Calendar, arising from the scantiness of the information handed down to us by the ancient writers, Censorinus and Geminus, on the subject. The following is, substantially, the view given by Ideler (i., 325, *sqq.*), and which is now generally accepted. The number of days in the cycle of 19 years was assumed to be 6940. If all the 235 months were *full* (= 30 days), the number of days would amount to 7050, being 110 too many. In order, therefore, to get rid of this surplus number, he made 110 of the 235 months *hollow* (= 29 days) (1). And, further, with a view to distribute these hollow months equally throughout the cycle, he divided the 7050 days by 110. The nearest integer quotient is 64; denoting that every 64th day, reckoning from the beginning, must be dropped out, or *exempted* (ἡμέραι ἐξαιρέσιμοι) (2); and as the ratio of 125 to 110 is nearly the same as that of 17 to 15, it follows that in every 32 months there must be 17 full and 15 hollow ones. Hence Meton's Canon or Table began with a full month; then followed full and hollow months, alternately, eight times in succession; after which came a full month, followed by a series of seven full and hollow months, alternately; and so on throughout, to the end of the cycle (3).

(1). The Calendar months of 30 days were called *full* (πλήρεις, *pleni*); those of 29 days were called *hollow* (κοῖλοι, *cavi*).

(2). Dodwell maintains that every 63rd day was *exemptive;* and this seems to have been the opinion of Delambre also, who says that 6940 was divided by 110, giving $63\frac{1}{11}$.

(3). See Meton's Canon, as given by *Ideler*, i., 383.

48. The nineteen-year Lunar Cycle, as arranged by Meton, was composed of 12 *Common* Lunar years, each consisting of 12 Lunar months, of 30 and 29 days alternately. Eight of those 12 years contained 354 days, and the other four 355. The remaining 7 years were *Embolismic* (ἐμβολιμαῖος, ἐμβάλλω) or intercalary years, consisting, each, of 13 months, and containing 384 days. There is a difference of opinion as to the position of the Embolismic years in the cycle. According to Petavius (Doctr. Temp. ii., 13), they were the 3rd, 6th, 8th, 11th, 14th, 17th, 19th (1). Ideler (i., 383) gives the 3rd, 5th, 8th, 11th, 13th, 16th, 19th. The Metonic *Calendar*, based on this cycle, was published at Athens in Olymp. 86, 4 (B. C. 433). The epoch, or starting point, of the cycle itself was July 15, B. C. 432, Ol. 87, 1 (2). As Meton made the 19 Solar years to contain 6940 days, it follows that the length of the Solar year was assumed by him to

be $365\frac{5}{19}$ days. This was $\frac{1}{76}$th greater than $365\frac{1}{4}$ days, a more approximate length, soon after, if not already, ascertained. This excess would amount to a whole day in 76 years; in other words, the Metonic cycle would be a day wrong at the end of that time, and would require to be corrected by *dropping* a day; that is to say, by making every fourth cycle to consist of 6939 days, instead of 6940. In this way, four Metonic cycles would contain 27,759 days, instead of 27,760. This correction was made, a century after Meton's time, by Callippus, of Cyzicus, in Ol. 110 (B. C. 340). The *Callippic period* (3), accordingly, consisted of three consecutive Metonic cycles of 6940 days each, *plus* a fourth cycle of 6939 days; making in all 76 years, each $365\frac{1}{4}$ days long. This was a very famous period in ancient Chronology, and was termed ἑκκαιεβδομηκονταετηρίς. About 200 years later, Hipparchus, the great astronomer, found that the Callippic year of $365\frac{1}{4}$ days was about $\frac{1}{300}$th of a day too long. He, therefore, proposed to omit one day at the end of every 304th year, that is, at the end of every 4th Callippic period (74 × 4). Thus, the length of Hipparchus' period (called by Censorinus *Annus Hipparchi*) was 111,035 days (4).

(1). The only reason why Petavius adopted this order seems to have been that it was so in the nineteen-year cycle of the Jews, who probably borrowed their cycle from the Metonic. Geminus throws no light on this question. Vid. *Ideler*, i., 542; ii., 237.

(2). Vid. *Ideler*, i., 326, 329.

(3). The proper distinction (which, however, is frequently neglected) between a *cycle* and a *period* is this:—A cycle is a recurring series of years, at the end of which certain phenomena, or time-relations, repeat themselves. A period contains two or more cycles. Thus, we speak of the Metonic cycle, and of the Callippic period. So, in like manner, we speak of the *Victorian* period of 532 years, arising from the combination of the Solar Cycle of 28 years, and the Lunar Cycle of 19 years; and of the *Julian period* of 7980 years, arising from the continued product of the Solar, Lunar, and Induction Cycles.

(4). Callippus' period was a nearer approximation than Meton's Cycle to the mean motion of the Moon as well as of the Sun. For 27,759 divided by 940 (= 235 × 4) gives for the synodic Lunation $29^d\ 12^h\ 44^m\ 25\frac{1}{2}^s$, which is only about 22 seconds too much.

Hipparchus' period of 111,035 days, divided by 304, gives for the length of the Solar year $365^d\ 5^h\ 55^m\ 15{\cdot}8^s$, which differs from the true by about $6\frac{1}{2}$ minutes. Again, 111,035, divided by 3760 (the number of Lunations in 304 years), gives for the length of the synodic Lunar month $29^d\ 12^h\ 44^m\ 2\frac{1}{2}^s$, differing from the true length about half a second. (Art. 40.)

49. There can be no doubt that after, if not before, the Nicene Council, the Alexandrian astronomers made use of the Metonic Cycle for the determination of Easter. But it does not appear that, even at Alexandria, this cycle was employed for that purpose earlier than the middle of the third century A. D. (1). So far as we know, the first person who so applied it was Anatolius, a native of Alexandria, who became Bishop of Laodicea, *circ.* A. D. 270 (2). When, and through what agency, the *nineteen-year* cycle was com-

pletely developed into an Easter cycle cannot with certainty be determined. According to the express testimony of Jerome and Bede, Eusebius, the celebrated Bishop of Cæsarea, had the chief part in the matter (3). However this may be, the nineteen-year cycle gradually displaced all the other cycles hitherto employed, and was exclusively used in the computation of Easter. In its adaptation to the Christian Calendar the Metonic Cycle underwent certain modifications which I now proceed to describe.

(1). Eusebius (Lib. vii. c. 20) mentions two Festal Epistles (ἐπιστολαὶ ἑορταστικαὶ) of Dionysius, Bishop of Alexandria (A. D. 248-265), in one of which he used the ὀκταετηρίς, or eight-year cycle.

(2). Vid. *Ideler*, II. 226. It is very remarkable that Epiphanius, who lived a century later, while he mentions and explains the eight-year cycle (*octaeteris*), does not say a word of the much more perfect nineteen-year cycle.

(3). Jerome (De Viris Illustr. c. 61) says, *Hippolytus* xvi. *annorum circulum, quem Græci* ἑκκαιδεκαετηρίδα *vocant, reperit, et Eusebio, qui super Pascha decem et novem annorum circulum, id est* ἐννεακαιδεκαετηρίδα, *composuit, occasionem dedit.*

And Beda (De Temp. Rat. c. 42) says, *Decemnovennalis circuli ordinem primus Eusebius, Cæsareæ Palestinæ Episcopus, ob quartasdecimas Lunas Festi Paschalis ipsumque diem Paschæ inveniendum, composuit.*

50. In the first place, then, the Callippic or Julian year of $365\frac{1}{4}$ days was adopted, and the Metonic Cycle was assumed to be rigidly exact; *i. e.*, that 235 Lunations are exactly equal to 19 Julian years (1). Now, $19 \times 365\frac{1}{4}^{d} = 6939\frac{3}{4}^{d}$; and this number divided by 235 gives $29^{d}\ 12^{h}\ 44^{m}\ 25\frac{1}{2}^{s}$ for the mean length of a Lunation. But, as fractional parts of a day cannot be employed in a Civil Calendar, the Lunations were taken to be 30 and 29 days, alternately; and were so arranged as that 235 of them should fill up the whole space of 19 Solar years. The mode in which this was effected was, in principle, the same as in the Metonic Cycle. The number of *complete* Lunar months (consisting of 30 and 29 days alternately) in a Solar year is 12; and, therefore, in 19 years, 228. These 228 months (30 and 29 days, alternately) were called *Common* months. The remaining 7 months (to complete 235) were called *Embolismic*, or intercalated months. Six of them consisted of 30 days each, and one of 29 days. They were called *Embolismic*, because those consisting of 30 days were *inserted* (ἐμβάλλω) among the Common months, at certain intervals, while the one of 29 days was *annexed* at the end of the entire series, and so was the last of the 235 (2). Hence, in this arrangement there are 120 *full* months, and 115 *hollow*, instead of Meton's 125 and 110.

Thus, there were 228 *Common* months $(114 \times 30 + 114 \times 29) = 6726$ days.

6	*Embolismic*	„	(30×6)	=	180	„
1	„	„	annexed at end,	=	29	„
235	Calendar months,	. . . .		=	6935	„ (3).

This is the number of days in 19 *Common* years (365 d.), and falls short of the number of days in 19 Julian years ($365\frac{1}{4}$) by $4\frac{3}{4}$ days. These deficient days were made up by adding, every Leap-year, one day to the Calendar Lunation in which the Bissextile day (24th) of February occurred, and so making the Lunation to consist of 31 or 30 days, according as the ordinary Lunation consisted of 30 or 29 days (4). But in 19 Julian years there may be four or five Leap-years; *five*, if the 1st, 2nd, or 3rd, year of the cycle be Leap-year; *four*, if the 4th year be Leap-year. In the former case the number of days in the cycle will be 6940 (6935 + 5)—6 hours *longer* than 19 Julian years; in the latter case, the number will be 6939 (6935 + 4)—18 hours *shorter* than 19 Julian years. The mean length of the cycle is, therefore, $\frac{6940 \times 3 + 6939}{4} = 6939\frac{3}{4}$ days; agreeing exactly with 19 Julian years. It was only at the expiration of every four periods of 19 years, or 76 years, that there was an exact coincidence between the Calendar Lunar Cycle and the Julian year. For in 76 years there are exactly 19 Leap-years $\left(\frac{76}{4} = 19\right)$; and in that same time, the $4\frac{3}{4}$ defective days amount to exactly 19 days; so that the Calendar Lunar Cycle exactly terminates with the close of every 76th Julian year (5). We thus meet again the famous Callippic period (Art. 48).

(1). The exact length of 235 mean Lunations, each 29·530588715 days, = 6939·688348025 days; and 19 Julian years = 6939·75 days; which exceeds the former by ·061651975^d = 1^h 28^m 47^s; that is to say, 235 mean Lunations are about $1\frac{1}{2}$ hours shorter than 19 Julian years.

(2). The term *Embolismic* properly applies only to the 6 *inserted* months. All the months of 30 days still continued to be called *full* (*pleni*), and those of 29 days *hollow* (*cavi*).

(3). The same result comes out in the following way:—$29\frac{1}{2} \times 235 = 6932\frac{1}{2}$ days. But the 6 Embolismic months of 30 days, each, add 3 days to this sum, while the last Embolismic month, of 29 days, takes away half a day. Hence the total sum is 6935.

(4). In a Bissextile year, the Lunation which contains the intercalary day—that is, the day interposed between February 23 and February 24 (Art. 25)—receives the addition of one day. Now, the first, or January, Lunation each year being always one of 30 days, the second Lunation is 29 days, the third 30, and so on. Hence all the Lunations that begin in January and *end* in February contain 29 days; and, therefore, all of them that *include* the intercalary day are reckoned, in Bissextile years, as containing 30 days. This takes place in those Bissextile years whose Golden Numbers are III., VI., XIV., XVII. (*vid.* the Old Church Calendar, Art. 51, *sqq.*). But in the case of Lunations beginning in February, and, therefore, ending in March, and which, consequently, contain 30 days, the Bissextile years corresponding to all the Golden Numbers in February that *precede* the intercalary day have their third Lunation 31 days. These Golden Numbers are all the nineteen, with the exception of the four above mentioned. A Lunation of 31 days is, as Delambre observes, a monstrosity in astronomy; but it could not be dispensed with. Cf. *Clavius*, p. 375.

(5). As there are nineteen Leap-years in four Julian Cycles of nineteen years each, there must be five

Leap-years in three of the cycles, and four in the remaining one. Hence, in every four such cycles, there will be one of 6939 days, which is too short by eighteen hours; and three of 1640 days, which, taken together, are too long by eighteen hours, so that the three exactly compensate the one.

51. We have now to consider how these 235 Calendar Lunations (228 Common and 7 Embolismic), and consisting, for the most part, of 30 and 29 days, alternately, were distributed in the Old Church Calendar throughout the successive years of the nineteen-year cycle (1). To avoid repetition, it must be borne in mind that when, in what follows, New Moons and Lunations are spoken of, Calendar New Moons, and Calendar Lunations (of 30 and 29 days), are meant, *not* real New Moons and astronomical Lunations. The Solar year is always assumed to be the *Common* year of 365 days, the Leap-years being taken into account in the way explained in Art 50.

The epoch, or commencement of the cycle, is arbitrary; that is to say, the cycle may commence with the year-day on which any one of the 235 New Moons falls. Let us suppose, then (the reason of the supposition we shall see presently), that a New Moon falls on January 1, in the *third* year of the cycle (see Table of Art. 52). This Lunation, containing 30 days, ends on January 30; and the following Lunation (29 days) begins on January 31. Reckoning on, 30 and 29 days alternately, we find that the other New Moons contained in this third year fall, respectively, on March 1, March 31, April 29, May 29, June 27, July 27, August 25, September 24, October 23, November 22, December 21. Accordingly, the Roman numeral III. is prefixed, in the Calendar, to each of these thirteen days, to indicate that, in *the third year of the Lunar Cycle, the New Moons fall on these days*. But it must be observed that there are only *twelve complete* Lunations this third year, the twelfth Lunation ending December 20, the 354th day of the year. The following Lunation (30 days) begins December 21; so that the Moon is 11 days old at the end of the year, and the Lunation ends on the 19th of the following January (11 + 19 = 30). This Lunation *is reckoned as the first Lunation* in the fourth year of the cycle, in accordance with a rule of the ancient Computists, which they expressed in the following verse:

In quo completur mensi Lunatio detur.

denoting that a Lunation is regarded as belonging to the month (and year) in which it *ends* (2). The first New Moon, this fourth year of the cycle, falls on January 20; and IV. is prefixed to that day and to the days on which the other Calendar New Moons (reckoned as before, 29 and 30 days, alternately) fall: viz., February 18, March 20, April 18, May 18, June 16, July 16, August 14, September 13, October 12, November 11, December 10;—all of them 11 days earlier than the New Moons of the third year. The twelfth Lunation this fourth year ends with December 9; and, therefore, the following Moon (which is the first of the fifth year of the cycle) is 22 days old at the end of the fourth year (3), and (the Lunation being one of 30 days) ends on the 8th of January. The first New Moon of the fifth year of the cycle falls on January 9; and V. is, accordingly, prefixed to it, and to all the other New Moon days that year: viz., February 7, March 9, April 7, May 7, June 5, July 5, August 3, September 2, October 2, October 31, November 30, December 29. This fifth year includes, therefore, thirteen Lunations (the first of them ending, as above explained, on January 8), and contains 384 days—in other words, V. is an *Embolismic year* (4). The thirteenth Lunation, which is one of 29 days (5), ends on December 28, leaving the next Moon three days old at the end of the year. Proceeding in the same way, VI. is attached to January 28, February 26, ... December 18, of the sixth year, a Common one. The Moon is 14 days old at the end of this year, and, accordingly, VII. is affixed to January 17 ($14 + 16 = 30$); and so on, down to December 7 of the seventh year, which is also a Common year, with only twelve Lunations. The Moon is 25 days old at the end of that year; and VIII. is accordingly affixed to January 6, ... December 26 of the eighth year. This eighth year is Embolismic, as it contains thirteen Lunations, leaving 5 days to be carried on to the next (6). Proceeding as before, we find the next (the third) Embolismic year to be the eleventh of the cycle; XI. is accordingly affixed to January 3, ... December 23. (7) The fourth Embolismic year is the thirteenth of the cycle. XIII. is affixed to January 11, ... December 31. (8) The sixteenth year of the cycle is Embolismic (the fifth). XVI. is affixed to January 8, . . . December 28. (9) The nineteenth and last year of the cycle is the sixth Embolismic year. XIX. is affixed to January 5, ... December 24. To this Embolismic year, as the last of the cycle, was assigned the Embolismic Lunation of 29 days (Art. 50); thus making this nineteenth year consist of only 383 days, whereas all the other Embolismic years contain 384 days. (10) The first New Moon of the cycle falls on January 23, and I. is affixed to January 23, ... December 13, so that the Moon is 18 days old at the end of the year. II. is accordingly prefixed to January 12, ... December 2, of the second year of the cycle; it is the seventh *and last Embolismic* year, containing 384 days. The thirteenth Lunation (30 days) ends the year, and completes the cycle, which again commences with III. affixed to January 1, &c.

(1). Two nineteen-year cycles are mentioned by ancient writers, which must be carefully distinguished—viz., the Alexandrian (*Christian*) Cycle, with which we are now concerned, to which the Golden Numbers are attached; and the *Jewish*. Dionysius, Exiguus, and Bede designate the former as *Cyclus decemnovennalis*, and the latter as *Cyclus lunaris*—a not very happy distinction, inasmuch as both are equally *Lunar* Cycles. The chief difference between them is that, in the Christian Easter Tables, the epoch, or beginning of the *Jewish Cycle*, is placed exactly three years later than the epoch of the Christian; in other words, the epoch of the *Cyclus lunaris* is on the 1st of January in the third year of the *Cyclus decemnov.* There is also a corresponding difference in the arrangement of the Embolismic years. In the Christian Cycle the Embolismic years (as shown in Art. 51) are

2nd, 5th, 8th, 11th, 13th, 16th, 19th;

in the Jewish Cycle they are

3rd, 6th, 8th, 11th, 14th, 17th, 19th.—

Ideler, ii. pp. 273-371; i. p. 542.

Petavius (Doctr. Temp. vi. 5), and many others of the older Chronologists, held that the Romans also had, from the time of Julius Cæsar, who first introduced it, a nineteen-year cycle, like that to which Dionysius and Bede give the appellation *lunaris*, in contradistinction to the Alexandrian. And, upon that supposition, Petavius draws up a perpetual Julian Calendar, differing from the Alexandrian Church Calendar (Art. 58, *sqq.*) only in this, that the Golden Numbers are throughout two units less: *e.g.*, while in the (Alexandrian) Church Calendar III. is affixed to January 1, in the so-called Roman Calendar I. is affixed. But there is no evidence to prove that the Latin Church made use of a nineteen-year cycle before their adoption of the Alexandrian Cycle; while, on the other hand, there is evidence to show that the cycle actually employed by the early Latin Church was the eighty-four-year cycle.—*Ideler*, ii. 241.

(2). The reason of this rule is thus given by *Clavius*, cap. xvii. § 1:—When the New Moon falls on January 1, it ends on January 30, and, in this case, there is no doubt that the first Lunation of the year belongs to January. The second Lunation, beginning with January 31, ends with last day of February, and is, therefore, *propter majorem partem*, to be assigned to it. The third Lunation is all contained in March. The fourth Lunation begins on March 31, and ends April 28, and is (as in February) assigned to April. And, following the same analogy, throughout the whole year the Lunations are assigned to the months in which they *end*, though they may have begun, and the greater part of them be contained, in the preceding months. And since, in a *Common* Lunar year, there are only twelve Lunations, the next Lunation (which in the case before us *begins* on December 21, and *ends* January 19) is to be assigned to the following January. The same holds also in the case of an *Embolismic* year, containing thirteen Lunations. The year *ends* in December; and the following Lunation, whether it begins in December or on January 1, is assigned to January. Hence it follows that, except in the single case where the New Moon falls on January 1, the *first* Lunation of every year in the cycle *begins* in December.

(3). Let n be any number in the series 0, 1, 2, 3, . . . 18, corresponding respectively to the Golden Numbers III., IV., V., VI., ... II.: then the number of days old the Moon is at the beginning of any year of the cycle may be found by multiplying n by 11, and dividing the product by 30; the *remainder* after the division will be the required age of the Moon. This Rule, expressed in the notation already used, is

$$\text{Moon's age} = \left(\frac{n \times 11}{30}\right)_r.$$

Ex. 1. Required the Moon's age on January 1, in the third year of the Lunar Cycle.

Here, $n = 0$; therefore Moon's age $= 0$.

Ex. 2. Required the Moon's age on January 1, in the sixth year of the cycle.

Here we have

$$\left(\frac{3 \times 11}{30}\right)_r = 3.$$—*Vid.* below, Art. 63.

(4). Since the Lunations are assigned to the months in which they *end*, so that every Lunation begin-

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ning in December, and *ending* on any day of January, even the 1st, must be assigned to January, and reckoned as the first Lunation of the year. It may happen in some cases that, *after the first Lunation* of a year, there will still remain a sufficient number of days (354 or 353) to complete twelve more Lunations before arriving at the first Lunation of the following year. In such a case, the year will contain thirteen Lunations, and will be an *Embolismic* year, containing 384 or 383 days. But if, after the first Lunation, there be only *eleven whole* Lunations remaining, the year will be a *Common one.* The necessary and sufficient condition for an Embolismic year may, therefore, be thus stated:—*The first Lunation of the year must end with some one of the first eleven days of January* (vid. *Clav.*, pp. 369, 372).

What has been just said also explains the apparent paradox that a year of 365 days may include 384 days. It does not, of course, actually contain so many *days*, but it includes *Lunations* which amount to that number of days.

(5). The ancient Computists strictly adhered to the *Rule of making the first Lunation, every year of the cycle (whether Common or Embolismic), to contain* 30 *days.* The rule of alternation (30 and 29 days) was adhered to as closely as possible. Hence it follows that, in a *Common* year, the twelfth and last Lunation consists of 29 days; and if all the years were Common, the alternation would always be maintained. But in an *Embolismic* year, of thirteen Lunations, the thirteenth and last Lunation would contain (if the alternation was kept up throughout) 30 days; and thus the alternation with the next year would be interrupted. The Computists were unwilling to disturb the alternation at this point of junction of two successive years, and, therefore, with two exceptions (fourth and seventh Embolism), they reduced the thirteenth Lunation to 29 days. But the day so taken away was added to one of the regular twenty-nine-day Lunations preceding, so that the total number of days in the thirteen Lunations remained unchanged. This change of 29 to 30 days also, of course, interrupted the alternation; but the Computists preferred that the interruption should take place anywhere rather than at the junction of two years. For example, in the case before us (year 5 of the cycle), if the rule of alternation had been strictly adhered to, V. should have been affixed to October 1 and November 29, instead of to October 2 and November 30, to which the Computists have affixed it. But the reason they did so was, that they added the day taken from the thirteenth Lunation to the tenth (29 days), thus making the tenth Lunation end with October 1, instead of September 30, and, consequently, transferring V. from October 1 to October 2. Hence the ninth and tenth Lunations consisted each of 30 days. The eleventh, twelfth, and thirteenth Lunations they then changed to 29, 30, and 29 days (instead of 30, 29, 30), to preserve the alternation. It is easy to see that this affected the place of V. only with respect to the eleventh Lunation (November 30, instead of November 29).—*Vid.* the Table.

(6). *Vid.* the last Note. In this case the fourth Lunation (29 days) got the day deducted from the thirteenth: and so was made in effect the Embolismic month. In this year, therefore, two Lunations of 30 days each (the third and fourth) come together.

(7). In this eleventh year the second Lunation (29 days) borrows a day from the Embolismic Lunation. Hence, in a Bissextile year, whose Golden Number is XI., and whose February Lunation (the third) must be increased by one day (Art. 51), it happens that the first *four* Lunations contain each 30 days. —*Clavius*, p. 375.

(8). This thirteenth year and the second year are the only Embolismic years of which the thirteenth Lunation is allowed to retain its 30 days.

(9). As in the case of the first Embolism, the tenth Lunation borrowed a day from the thirteenth, and so was the Embolismic month.

(10). Strictly speaking, the Embolismic Lunation of 29 days ought to belong to the seventh and last Embolismic year (Art. 50), viz., the second year of the cycle. But the ancient Computists preferred to make it belong to the nineteenth and last year of the cycle. They accordingly made three Lunations of 29 days each come together, viz., seventh, eighth, and ninth.—*Vid.* Clavius, cap. xii., 13, cap. xvii., 15, p. 375).

52. The following Table, taken from Clavius, p. 376, should be looked at while reading the last Article, and the Notes appended to it.

Golden Numbers.	Beginning of 1st Lunation, both in Common and Embolismic years.	End of 12th Lunation in both.	End of 13th Lunation in Embolismic year.	Number and Order of Lunations throughout the year, with their Respective Number of Days.												
				1st Lunation.	2nd "	3rd "	4th "	5th "	6th "	7th "	8th "	9th "	10th "	11th "	12th "	13th "
III.	1 Jan.	20 Dec.		30	29	30	29	30	29	30	29	30	29	30	29	
IV.	21 Dec.	9 ,,		30	29	30	29	30	29	30	29	30	29	30	29	
* V.	10 ,,	29 Nov.	28 Dec.	30	29	30	29	30	29	30	29	30	30	29	30	29
VI.	29 ,,	17 Dec.		30	29	30	29	30	29	30	29	30	29	30	29	
VII.	18 ,,	6 ,,		30	29	30	29	30	29	30	29	30	29	30	29	
* VIII.	7 ,,	26 Nov.	25 Dec.	30	29	30	30	29	30	29	30	29	30	29	30	29
IX.	26 ,,	14 Dec.		30	29	30	29	30	29	30	29	30	29	30	29	
X.	15 ,,	3 ,,		30	29	30	29	30	29	30	29	30	29	30	29	
* XI.	4 ,,	23 Nov.	22 Dec.	30	30	29	30	29	30	29	30	29	30	29	30	29
XII.	23 ,,	11 Dec.		30	29	30	29	30	29	30	29	30	29	30	29	
* XIII.	12 ,,	30 Nov.	30 Dec.	30	29	30	29	30	29	30	29	30	29	30	29	30
XIV.	31 ,,	19 Dec.		30	29	30	29	30	29	30	29	30	29	30	29	
XV.	20 ,,	8 ,,		30	29	30	29	30	29	30	29	30	29	30	29	
* XVI.	9 ,,	28 Nov.	27 Dec.	30	29	30	29	30	29	30	29	30	30	29	30	29
XVII.	28 ,,	16 Dec.		30	29	30	29	30	29	30	29	30	29	30	29	
XVIII.	17 ,,	5 ,,		30	29	30	29	30	29	30	29	30	29	30	29	
* XIX.	6 ,,	24 Nov.	23 Dec.	30	29	30	30	29	30	29	29	29	30	29	30	29
I.	24 ,,	12 Dec.		30	29	30	29	30	29	30	29	30	29	30	29	
* II.	13 ,,	1 ,,	31 Dec.	30	29	30	29	30	29	30	29	30	29	30	29	30

Note.—The Embolismic years are marked with an asterisk.

53. The explanation given in Art. 51, of the way in which the framers of the ancient Church Calendar distributed the New Moons, or (which is the same thing) the Golden Numbers indicating those Moons, over the successive years of the nineteen-year cycle (or Lunar Cycle, as I shall henceforth call it) was based, as I have said, on the supposition that a New Moon fell on the 1st of January, in the *third* year of the cycle, Golden Number III. The supposition was made because it represented the actual fact. The epoch of the Lunar Cycle is (as I have said, Art. 51) arbitrary. The Alexandrian astronomers assumed it to be March 23, A. D. 323. And the reason why they chose this epoch was, that by their calculation the Vernal Equinox took place on that day in that year, and moreover there was a New Moon that same day. This coincidence seems to

have been regarded as a sufficiently remarkable epoch from which to begin the cycle, and accordingly they affixed G. N. I. to March 23. Reckoning back 59 days (30 and 29) from this, the same G. N. I. was affixed to February 21 and January 23: and reckoning forwards 29 and 30 days alternately, it was also affixed to April 21, May 21, &c., to the end of the year. The next year (A. D. 324) all the New Moons fell 11 days earlier, and G. N. II. was accordingly affixed to January 12, February 10, March 12, &c. The next or third year of the cycle, the New Moons fell another 11 days earlier; accordingly, G. N. III. was affixed to January 1, January 31, March 1, &c. And thus, a New Moon fell on the first day of the year of the famous Nicene Council (325), which may have been an additional reason for choosing the above epoch for the cycle. In this way, following the course pointed out already in Art. 51, throughout the whole cycle of nineteen years, we can exhibit the whole scheme of the *Old Church Calendar*. This Calendar is also known as the *perpetual Julian Calendar;* because if the data on which it was constructed were *strictly accurate* (and not merely approximate, as they in fact were), this Calendar, constructed for one Lunar Cycle, would apply to all preceding and subsequent cycles (Art. 47), and so would suffice to find all the Calendar New Moons—including, of course, the Paschal New Moons—for ever. There is some slight diversity as to the arrangement of the Golden Numbers in the Calendar, as it is given in the standard Chronological books. I have followed Clavius, p. 108. The Letters in the columns headed L denote the Calendar Letters (Art. 23) belonging to the days of each month of the year; and the Figures in the columns headed N are the Golden Numbers indicating the Calendar *New* Moons, throughout a complete cycle of 19 years.

It will be observed that this following Table is in fact nothing more or less than the application of the foregoing Table (Art. 52) to the Common year of 365 days.

THE ANCIENT CHURCH CALENDAR.

Days of Month.	Jan.		Feb.		Mar.		Apr.		May.		June.		July.		Aug.		Sept.		Oct.		Nov.		Dec.		Date of Month.
	L.	N.	L.	N.	L.	N.	L.	N.	L.	N.	L.	N.	L.	N.	L.	N.	L.	N.	L.	N.	L.	N.	L.	N.	
1	A	III.	D		D	III.	G		B	XI.	E		G	XIX.	C	VIII.	F	XVI.	A	XVI.	D		F	XIII.	1
2	B		E	XI.	E		A	XI.	C		F	XIX.	A	VIII.	D	XVI.	G	V.	B	V.	E	XIII.	G	II.	2
3	C	XI.	F	XIX.	F	XI.	B		D	XIX.	G	VIII.	B		E	V.	A		C	XIII.	F	II.	A		3
4	D		G	VIII.	G		C	XIX.	E	VIII.	A	XVI.	C	XVI.	F		B	XIII.	D	II.	G		B	X.	4
5	E	XIX.	A		A	XIX.	D	VIII.	F		B	V.	D	V.	G	XIII.	C	II.	E		A	X.	C		5
6	F	VIII.	B	XVI.	B	VIII.	E	XVI.	G	XVI.	C		E		A	II.	D		F	X.	B		D	XVIII.	6
7	G		C	V.	C		F	V.	A	V.	D	XIII.	F	XIII.	B		E	X.	G		C	XVIII.	E	VII.	7
8	A	XVI.	D		D	XVI.	G		B		E	II.	G	II.	C	X.	F		A	XVIII.	D	VII.	F		8
9	B	V.	E	XIII.	E	V.	A	XIII.	C	XIII.	F		A		D		G	XVIII.	B	VII.	E		G	XV.	9
10	C		F	II.	F		B	II.	D	II.	G	X.	B	X.	E	XVIII.	A	VII.	C		F	XV.	A	IV.	10
11	D	XIII.	G		G	XIII.	C		E		A		C		F	VII.	B		D	XV.	G	IV.	B		11
12	E	II.	A	X.	A	II.	D	X.	F	X.	B	XVIII.	D	XVIII.	G		C	XV.	E	IV.	A		C	XII.	12
13	F		B		B		E		G		C	VII.	E	VII.	A	XV.	D	IV.	F		B	XII.	D	I.	13
14	G	X.	C	XVIII.	C	X.	F	XVIII.	A	XVIII.	D		F		B	IV.	E		G	XII.	C	I.	E		14
15	A		D	VII.	D		G	VII.	B	VII.	E	XV.	G	XV.	C		F	XII.	A	I.	D		F	IX.	15
16	B	XVIII.	E		E	XVIII.	A		C		F	IV.	A	IV.	D	XII.	G	I.	B		E	IX.	G		16
17	C	VII.	F	XV.	F	VII.	B	XV.	D	XV.	G		B		E	I.	A		C	IX.	F		A	XVII.	17
18	D		G	IV.	G		C	IV.	E	IV.	A	XII.	C	XII.	F		B	IX.	D		G	XVII.	B	VI.	18
19	E	XV.	A		A	XV.	D		F		B	I.	D	I.	G	IX.	C		E	XVII.	A	VI.	C		19
20	F	IV.	B	XII.	B	IV.	E	XII.	G	XII.	C		E		A		D	XVII.	F	VI.	B		D	XIV.	20
21	G		C	I.	C		F	I.	A	I.	D	IX.	F	IX.	B	XVII.	E	VI.	G		C	XIV.	E	III.	21
22	A	XII.	D		D	XII.	G		B		E		G		C	VI.	F		A	XIV.	D	III.	F		22
23	B	I.	E	IX.	E	I.	A	IX.	C	IX.	F	XVII.	A	XVII.	D		G	XIV.	B	III.	E		G	XI.	23
24	C		F		F		B		D		G	VI.	B	VI.	E	XIV.	A	III.	C		F	XI.	A	XIX.	24
25	D	IX.	G	XVII.	G	IX.	C	XVII.	E	XVII.	A		C		F	III.	B		D	XI.	G	XIX.	B		25
26	E		A	VI.	A		D	VI.	F	VI.	B	XIV.	D	XIV.	G		C	XI.	E	XIX.	A		C	VIII.	26
27	F	XVII.	B		B	XVII.	E		G		C	III.	E	III.	A	XI.	D	XIX.	F		B	VIII.	D		27
28	G	VI.	C	XIV.	C	VI.	F	XIV.	A	XIV.	D		F		B	XIX.	E		G	VIII.	C		E	XVI.	28
29	A				D		G	III.	B	III.	E	XI.	G	XI.	C		F	VIII.	A		D	XVI.	F	V.	29
30	B	XIV.			E	XIV.	A		C		F		A	XIX.	D	VIII.	G		B	XVI.	E	V.	G		30
31	C	III.			F	III.			D	XI.			B		E				C	V.			A	XIII.	31

54. From what has been said in Art. 51, combined with the last two Tables, the following, among other less important results, may be deduced.

1°. That in every year of the Lunar Cycle, except the third (G. N. III.) the *first* Lunation of the year begins in December; while in the third year it begins on Jan. 1. And all the *first* Lunations, without exception, contain 30 days. (Table, Art. 52; and Note 5, Art. 51).

2°. In Common Lunar years the Lunations succeed each other in regular alternations of 30 and 29 days. This order is unavoidably interrupted in the Embolismic years, which gave rise to six pair of *consecutive* Lunations of 30 days, with one triplet of 29 days. (Arts. 51, 52).

3°. If G. N. XI. belong to *Bissextile* year, there will be four consecutive Lunations of 30 days each. This arises from the fact that the *third* Lunation (29 days) of G. N. XI. includes the Bissextile day, and, therefore, must be increased by 1. (Art. 51).

4°. All the other G. Nos., except VI., XIV., XVII., have Lunations of 30 days, beginning in February, and including the Bissextile day: therefore, in Bissextile years, these Lunations are of 31 days. (Art. 51).

5°. In the Table (Art 52) *seven* of the G. Nos. occur 13 times; viz., III., XI., XIX., VIII., XVI., V., XIII. All the rest occur but 12 times. These seven are found in the *first* 11 days of January and March, and in the *last* 11 days of December. In no other month are they found together in their order as above. In February, G. N. III. does not appear at all; and, therefore, in that month there are but 18 G. Nos. In April, June, September, and November, all the G. Nos. occur once. In the other months, except October, one of them occurs twice, and in October, two (viz., XVI. and V.) occur twice; thus giving 21 G. Nos. in that month. In January and March the G. Nos. fall on the same day of the month. It is also to be observed that 4 G. Nos. are crowded together in the beginning of April, by the moving up one place of XVI. and V. This was done in order that the Lunations beginning on the 8th and 9th of March should consist of 29 days only; the Computists having adopted the rule that *all* the *Paschal* Lunations should consist of 29 days. (*Vid.* Art. 59).

6°. The whole series of 19 G. Nos., as they follow each other, starting from any assumed No. in the Calendar (*e.g.* III.), is generated by adding 8 to the *preceding* G. N., and casting away 19 whenever the resulting sum exceeds 19; or (which amounts to the same) by subtracting 11 from the preceding G. N., previously adding 19 to the preceding, if it be less than 12. For example, XI. follows next after III., and = III + 8; XIX. follows XI., and = XI. + 8; VIII. follows XIX., and = XIX. + 8 − 19. Again, VIII. = XIX. − 11; XIX. = XI. + 19 − 11.

Similarly, any G. N. is formed from the next *succeeding* G. N. by subtracting 8, or by adding 11. For example, III. = XI. − 8; XI. = XIX. − 8; XIX. = XI. + 8.

Generally, the law of successive formation of the G. Nos. may be thus expressed. Let N be any G. N., and N' the next succeeding one; then

$$N' = \left(\frac{N+8}{19}\right)_r \text{ (1); when } r = 0,\ N = \text{XI, and } N' = \text{XIX.}$$

$$N = \left(\frac{N'+11}{19}\right)_r \text{ (2); when } r = 0,\ N' = \text{VIII., and } N = \text{XIX.}$$

The *reason* of this law of formation will be given hereafter (Art. 55) (¹).

7°. A *lesser* G. N. *following a greater* is, generally speaking, placed *immediately* after it, without any interval; *e. g.* XIX. and VIII.; XVI. and V.; XIII. and II. On the contrary, when a lesser number *precedes* a greater, it is generally separated from it by an interval of one place; *e.g.* III. and XI.; XI. and XIX.; VIII. and XVI. (²) (*Vid.* Art. 55).

8°. The Golden Number of any year of the cycle usually precedes the Golden Number of the year immediately before it by 11 days, in Table, Art. 53; *e. g.* III. precedes II. by 11 days; and II. precedes I. by 11 days; and so does IV. precede III. by the same number of days. This arises from the fact that the Solar year (365 days) exceeds the Lunar (354) by 11 days; so that the New Moons of a succeeding year fall 11 days earlier, each month, than those of the year immediately preceding. Very rarely the greater G. N. exceeds the next by *ten* days; and still more rarely by *twelve*. Any succeeding G. N. is separated from that immediately preceding by 19 days; *e. g.* IV. is 19 days below III. This follows from what has just been said; IV. precedes III. by 11, and therefore the *next* IV. falls 19 days after III.

9°. It must be carefully observed that during the whole course of the Lunar Cycle no two or more New Moons fall, or can fall, on the same day of the month; in other words, each New Moon of the whole 235 has its own day of the year appropriated to itself, and unshared by any other. This Rule was rigorously adhered to by the old Computists. Indeed, from the very nature of the cycle—namely, that it is only at the *end* of 19 years, and of 235 actual Lunations, that even an approximation takes place to a common measure of the Solar year and a Lunation—it necessarily follows that no two *actual* New Moons can fall on the same day of the month during the course of the cycle. The same is true of the *mean* New Moons. The only instance in which any of the old Computists permitted the occurrence of two *Calendar* Moons on the same day was in the case of the G. Nos. XIII. and II., both of which Campanus and some others affixed to Dec. 3rd. But most of the Computists rejected this, as inconsistent with the fundamental principle of the cycle, and they, accordingly, affixed XIII. to Dec. 1, and II. to Dec. 2 (³).

(1). We have seen (5°) that seven of the G. Nos. are repeated eighteen times in the course of the cycle, and that these Numbers are found on the first eleven days of January, and the last eleven days of December. Consequently, the law of successive generation of the G. Nos. by the addition of 8 to each *preceding* No. does not hold in passing from the end of December to the beginning of January; XIII. is affixed to December 31, from which III. (affixed to Jan. 1) does not follow by adding 8. In fact, the law is discontinued until we arrive at Jan. 12, whose G. N. II. is formed from XIII. by adding 8, and dropping 19.—*Clavius*, p. 112.

(2). The exceptions to the former part of this Rule are the following:—In February 3, XIX. follows XI. without any interval; in April 6, June 4, and August 2, XVI. follows *immediately* after VIII.; in October 3, and Dec. 1, XIII. follows *immediately* after V. These exceptions are caused by the rule of the alternate sequence of 30- and 29-day Lunations. Again, at the end of July, and in the following months to the end of the year, XIX. is moved *up* one place, thereby separating it one place from VIII., which, regularly, it ought to precede immediately; and, moreover, bringing it immediately after XI., whereas, regularly, there ought to be an interval of one place between them.—Vid. *Clavius*, cap. ix., § 5.

(3). Vid. *Clavius*, l. c.

55. It is easy to show that the New Moons indicated by the Golden Numbers in the Calendar (Art. 53) agree very nearly with the *mean* New Moons (Art. 40, Note); and also to perceive the reason of the law of formation of the successive Golden Numbers, as stated above (Art. 54, 6°, 7°).

I. In Eight Julian years of $365\frac{1}{4}$ days each, there are 2922 days. Again, 99 mean Lunations (each 29·530588 days) = $2923{\cdot}528283^d = 2923^d\ 12^h\ 40^m\ 43^s$; so that, at the end of eight years, the mean New Moons fall a little more than a day and a-half *later* than they did at the beginning. Suppose, then, a mean New Moon were to fall on Jan. 3, after the lapse of eight Julian years the mean New Moon would fall one and a-half days lower down. But the Calendar New Moon (G. N. XIX.) is affixed to Jan. 5; so that it differs from the place of the mean New Moon by less than half a day. Again, after the lapse of another eight years, the mean New Moon would fall about three days lower down than what it was at the beginning; that is to say, on Jan. 6; and we find G. N. VIII. affixed to this day; so that the mean and Calendar New Moon fall on the *same* day. In the same way, after a third period of eight years, the mean New Moon will fall about one and a-half days below Jan. 6, and, therefore, differ about half a day from the place of the Calendar New Moon (XVI.) affixed to Jan. 8. And at the end of a fourth period of eight years the mean New Moon will fall three days after the 6th, that is, on the 9th, where we also find G. N. V. The same is generally true of the other G. Nos., which occur together in *pairs*, each pair being separated from the next pair by an interval of one vacant place. A few of the G. Nos. interrupt this order, the interruption being required by the necessary equation of the Moon. Looking at the G. Nos. in January, for example, we see that XI., X., IX. are *isolated*, with an interval of one place before and after them.

II. We have, in what precedes, seen the reason of the law of formation, $N' = \left(\frac{N+8}{19}\right)_r$ (Art. 54, 6°); and that the Calendar New Moons agree very nearly with the mean New Moons throughout the cycle of nineteen years. Let us now consider the reason of the other law of formation, $N = \left(\frac{N' + 11}{19}\right)_r$.

In 11 Julian years there are $4017^d\ 18^h$: and 136 mean Lunations $= 136 \times 29{\cdot}530588^d = 4016^d\ 3^h\ 50^m\ 21^s$: so that after the lapse of 11 years the mean New Moons will fall one day and fourteen hours, or about one and a-half days, *earlier* than they did at the beginning of that period. Hence, suppose a mean New Moon to fall on Jan. 6 (G. N. VIII.), after the lapse of 11 years it will fall about one and a-half days *higher* up, that is between 5th and 4th. But we find G. N. XIX., which comes 11 years after VIII., affixed to Jan. 5; and, therefore, distant only about half-a-day from the place of the mean Moon. After the lapse of a second period of 11 years, the mean New Moon will fall about 3 days higher up than its original place, that is, it will fall on Jan. 3; where we find G. N. XI., which comes *eleven* years after G. N. XIX.: so that at the end of 22 years from the beginning, the mean and Calendar New Moons fall on the same day. In the same way it may be shown that any other group of three Calendar New Moons agrees very closely with the corresponding mean New Moons.

We have now seen that the Calendar New Moons of the Lunar Cycle very nearly agree throughout with the mean New Moons; and we have also seen the reason why, in general, any Golden Number is formed from the *preceding* one by the addition of 8; and from the *succeeding* one by the addition of 11. It was from this relation of the nineteen-year cycle to the numbers 8 and 11 that the ancient Computists resolved that cycle into an *ogdoad* and a *hendecad*. (1) It is obvious that, having the law of formation, we can derive the whole series of Golden Numbers from any given one.

(1). Thus, Dionysius Exiguus, in his *Epistola ad Bonifacium*, says, "Decemnovennalis Cyclus per ogdoadem et hendecadem in se revolvitur."—Vid. *Clavius*, cap. ix. §§ 7–9; *Ideler*, ii. 231.

56. Dionysius Exiguus adopted the Alexandrian epoch, or starting point, of the Lunar Cycle—viz., March 23, in the second year before the Nicene Synod (Art. 53). Reckoning backward from this, he found that a cycle *began* the *year before* the Christian

Era, as calculated by him. He, accordingly, substituted the year of Christ's birth for the Alexandrian epoch of the Lunar Cycle; or, which is the same thing, he took the year before the Christian Era as the first year of the Lunar Cycle. Hence we see the reason of the Common Rule for finding the Golden Number corresponding to any given year A. D.:—

"*Add* 1 *to the number of the year, and divide the sum by* 19: *if there be no remainder, the Golden Number is* 19; *if there be a remainder, it is the Golden Number required.*"

The Rule may be formulated thus:—Let x be the number of any year A. D.; and r the remainder after dividing $x + 1$ by 19; then

$$N = \left(\frac{x+1}{19}\right)_r.$$

Ex. 1. Find the Golden Number of A. D. 323.

Here $\left(\frac{324}{19}\right)_r = 1$, the required number. The quotient is 17, showing that 17 complete cycles had elapsed, and that the first year of the eighteenth cycle was current.

Ex. 2. Find the Golden Number of A. D. 322.

We have $\left(\frac{323}{19}\right)_r = 0$; denoting that 322 is the last year of a cycle (17th), and, therefore, that 19 was the current Golden Number.

Ex. 3. Find the Golden Number of A. D. 17.

We have $\left(\frac{18}{19}\right)_r = 18$.

57. It is easy to find the Golden Number (N') for any year (x') *Before Christ.*

As B. C. 1 was the first year of a cycle, so were also B. C. 20 (= 19 + 1); B. C. 39 (= 2 × 19 + 1); B. C. 58 (= 3 × 19 + 1); &c. But it is obvious that, if we subtract any year (x') from the year on which the cycle in which x' is found begins, we shall obtain the number of Golden Numbers *elapsed* before the beginning of x': consequently, to find the current Golden Number of x', we must add 1: that is to say,

$$N' = n \times 19 + 1 - x' + 1 = n \times 19 - x' + 2.$$

The Rule, then, for finding the Golden Number of any year (x') B. C. is this:—

Take the least multiple of 19 *that will give a number higher than the year number: to the difference between this and the year number add* 2: *this is the required year of the cycle.*

Ex. 1. Find the Golden Number of B. C. 26.

Here we have $n = 2$; $x' = 26$; therefore,

$$N' = 38 - 26 + 2 = 14.$$

Ex. 2. Find the Golden Number of B.C. 1.

$$N' = 19 - 1 + 2 = 20 = 1.$$

58. The formulæ given in Arts. 56 and 57 will find the Golden Number for any year between A.D. and B.C. It is, however, convenient to have Tables that will show the Golden Number on inspection, and without any calculation. The two following Tables answer this purpose.

TABLE I.

SHOWING THE GOLDEN NUMBER, WHICH IS THE SAME BOTH IN OLD AND NEW STYLE, FROM THE BIRTH OF CHRIST TO A.D. 9400.

Centurial Years.					Years from 0 to 100.																		
					0 19 38 57 76 95	1 20 39 58 77 96	2 21 40 59 78 97	3 22 41 60 79 98	4 23 42 61 80 99	5 24 43 62 81 100	6 25 44 63 82	7 26 45 64 83	8 27 46 65 84	9 28 47 66 85	10 29 48 67 86	11 30 49 68 87	12 31 50 69 88	13 32 51 70 89	14 33 52 71 90	15 34 53 72 91	16 35 54 73 92	17 36 55 74 93	18 37 56 75 94
					Golden Numbers.																		
00	19	38	57	76	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	20	39	58	77	6	7	8	9	10	11	12	13	14	15	16	17	18	19	1	2	3	4	5
2	21	40	59	78	11	12	13	14	15	16	17	18	19	1	2	3	4	5	6	7	8	9	10
3	22	41	60	79	16	17	18	19	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4	23	42	61	80	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	1
5	24	43	62	81	7	8	9	10	11	12	13	14	15	16	17	18	19	1	2	3	4	5	6
6	25	44	63	82	12	13	14	15	16	17	18	19	1	2	3	4	5	6	7	8	9	10	11
7	26	45	64	83	17	18	19	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
8	27	46	65	84	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	1	2
9	28	47	66	85	8	9	10	11	12	13	14	15	16	17	18	19	1	2	3	4	5	6	7
10	29	48	67	86	13	14	15	16	17	18	19	1	2	3	4	5	6	7	8	9	10	11	12
11	30	49	68	87	18	19	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
12	31	50	69	88	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	1	2	3
13	32	51	70	89	9	10	11	12	13	14	15	16	17	18	19	1	2	3	4	5	6	7	8
14	33	52	71	90	14	15	16	17	18	19	1	2	3	4	5	6	7	8	9	10	11	12	13
15	34	53	72	91	19	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
16	35	54	73	92	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	1	2	3	4
17	36	55	74	93	10	11	12	13	14	15	16	17	18	19	1	2	3	4	5	6	7	8	9
18	37	56	75	94	15	16	17	18	19	1	2	3	4	5	6	7	8	9	10	11	12	13	14

The construction of the above Table is as follows:—

In the five columns under "Centurial Years" are written, in order, the Centurial Figures of the years 0, 100, 200, 300, &c., up to 9400. In the six horizontal rows at the top are written the successive numbers from 0 (the year before the Christian Era, when a cycle of Golden Numbers began) up to 100. In the nineteen horizontal rows under "Golden Numbers," the Golden Numbers are written consecutively, beginning with 1 (corresponding with B.C. 1). Each of these rows differs from the preceding by 5 units, because in 100 years there are 5 cycles of Golden Numbers *plus* 5 ($100 = 19 \times 5 + 5$). Again, there are 19 centuries in each vertical column of *Centurial* years, because after the lapse of 19 centuries, the Golden Numbers of the *centuries* recur as before.

One or two examples will suffice to illustrate the use of this Table.

Ex. 1. Find the Golden Number of A. D. 323.

In the first centurial column find 3; and in the top columns find 13: at the intersection of the horizontal row of the first and the vertical row of the second, we find 1, which is the Golden Number required.

Ex. 2. Find the Golden Number of A. D. 1875.

The horizontal row of the centurial year 18 is intersected by the vertical row of 75 in the top columns at 14; which is, therefore, the Golden Number required.

It is obvious that this Table may be extended to any length, by simply writing additional centurial columns.

TABLE II.

To find the Golden Number for any year from the Birth of Christ to B.C. 4000.

Centurial Years.			Years from 1 to 100.																		
			1 20 39 58 77 96	2 21 40 59 78 97	3 22 41 60 79 98	4 23 42 61 80 99	5 24 43 62 81 100	6 25 44 63 82	7 26 45 64 83	8 27 46 65 84	9 28 47 66 85	10 29 48 67 86	11 30 49 68 87	12 31 50 69 88	13 32 51 70 89	14 33 52 71 90	15 34 53 72 91	16 35 54 73 92	17 36 55 74 93	18 37 56 75 94	19 38 57 76 95
			Golden Numbers.																		
00	19	38	1	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2
1	20	39	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	19	18	17	16
2	21	40	10	9	8	7	6	5	4	3	2	1	19	18	17	16	15	14	13	12	11
3	22		5	4	3	2	1	19	18	17	16	15	14	13	12	11	10	9	8	7	6
4	23		19	18	·17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
5	24		14	13	12	11	10	9	8	7	6	5	4	3	2	1	19	18	17	16	15
6	25		9	8	7	6	5	4	3	2	1	19	18	17	16	15	14	13	12	11	10
7	26		4	3	2	1	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5
8	27		18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	19
9	28		13	12	11	10	9	8	7	6	5	4	3	2	1	19	18	17	16	15	14
10	29		8	7	6	5	4	3	2	1	19	18	17	16	15	14	13	12	11	10	9
11	30		3	2	1	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4
12	31		17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	19	18
13	32		12	11	10	9	8	7	6	5	4	3	2	1	19	18	17	16	15	14	13
14	33		7	6	5	4	3	2	1	19	18	17	16	15	14	13	12	11	10	9	8
15	34		2	1	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3
16	35		16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	19	18	17
17	36		11	10	9	8	7	6	5	4	3	2	1	19	18	17	16	15	14	13	12
18	37		6	5	4	3	2	1	19	18	17	16	15	14	13	12	11	10	9	8	7

The construction of this Table is the same as that of Table I. The successive horizontal rows of Golden Numbers differ by *fourteen* units, instead of by *five*, as in Table I.; because in this case the Golden Numbers are written in the *reverse* order; and, therefore, the common difference of two consecutive rows is the complement of the former common difference (14 + 5 = 19).

The Table can, of course, be extended backwards to any number of centuries B.C.

Ex. 1. Find the G. N. of B.C. 2. The horizontal row of 00 (the first century B.C.) intersects the vertical column containing 2 (in the top series), in 19, which is, therefore, the G. N. required.

Ex. 2. Required the G. N. of B.C. 4004. The horizontal row containing 40 (in the third centurial column) intersects the vertical column containing 4 (in the top series), in 7, which is the required G. N.

59. Having now pointed out how the *Sundays* in any given year may be found; and, further, how the *Golden Number*, which determines the Calendar New Moons for any year, may be ascertained; and having shown how the nineteen Golden Numbers are distributed through the months of each year of the nineteen-year cycle, we have next to explain how these two necessary and sufficient elements—the Sunday Letter and the Golden Number—are applied to the determination of *Easter* Sunday. The Table, Art. 53, shows the arrangement of the Golden Numbers through all the months of the year. The question now is, within what portion of the year are the Golden Numbers which specially indicate Easter Sunday to be sought for; in other words, what are the *limits* of the Easter, or *Paschal, New Moons*, and on what days of the year can Easter Sunday fall? Now, by Rule III., Art. 41, the *earliest* day on which the fourteenth of the *Paschal* Moon can fall is the 21st of March. If the fourteenth day of the Moon fell earlier than the 21st, it was regarded as belonging, not to the *first* month, but to the *twelfth*, of the preceding year. Consequently, reckoning back 14 days from March 21, we find that the *earliest possible Paschal New Moon* falls on the 8th of March, whose Golden Number is XVI. As March 7 has no Golden Number affixed to it, no fourteenth Moon in the Old Calendar can fall on March 20. Golden Number VIII. is affixed to March 6, and the fourteenth day of that Calendar Moon falls on March 19. But this precedes the Equinox (March 21), and, therefore, the Golden Number that gives the earliest possible Paschal Moon is XVI. The *latest* Paschal New Moon, on the other hand, is that which falls on April 5 (Golden Number VIII.), because the whole cycle of the 19 Golden Numbers, reckoned from the 8th of March, inclusive, is then completed. And if another Paschal New Moon could fall on the 6th of April (Golden Number XVI.), there would be *two* Paschal New Moons, denoted by the same Golden Number XVI.; in other words, Easter would, in any year whose Golden Number is XVI., be celebrated *twice*, which is absurd; and the same is equally true of any other day after the 6th of April. Hence, the *latest* Paschal New Moon is that which falls on the 5th of April. The fourteenth day of that Moon is the 18th of April, the next Sunday *after* which will be Easter Day. But if the 18th be itself a *Sunday*, the next Sunday will, by Rule II., be Easter Day, April 25; which, therefore, is the *latest day on which Easter can possibly fall.*

Hence the extreme limits of the *Paschal New Moons* are March 8 (Golden Number XVI.), and April 5 (Golden Number VIII.): and the *extreme limits of Easter Day* itself are March 22 (the *earliest possible* day) and April 25 (the *latest possible* day); in other words, the time of Easter may vary to the extent of 35 days or 7 weeks.

It must be remarked that, while there are only 19 days (corresponding to the 19 Golden Numbers) on which the fourteenth day of the Moon can fall, yet Easter Day may itself fall on *any* of the 35 days within the above limits. This arises from the fact that any one of these days may be Sunday.

It is also to be observed that, as all the 19 Paschal Lunations consist severally of 29 days (*vid.* Art. 54), the whole cycle series of the 19 Paschal Golden Numbers, from XVI. (March 8) to VIII. (April 5), is contained within the space of 29 days. Of these 29 days only 19 have Golden Numbers affixed; so that 10 are vacant. In other words, on those days, in the Old Church Calendar, no Paschal New Moon could fall. In the Gregorian Calendar, as we shall see, this is otherwise; there may be 29 New Moons—24 in March and 5 in April (*Clavius*, cap. i. § 10).

From inspection of the Calendar, it further appears that when Easter falls on the earliest possible day (March 22), the Sunday Letter must be D; and when it falls on the latest possible day (April 25), the Sunday Letter must be C.

60. From what has just been said it appears that the portion of the Old Church Calendar specially applicable to the determination of Easter is contained within the limits of 35 days, counting from the earliest possible day (March 22) to the latest (April 25). The Table exhibiting the Paschal *Full* Moons (or fourteenth days, counting from the days of the New Moons, respectively, as indicated by the Golden Numbers) during this interval is called the *Old Paschal Table*, by means of which Easter was actually determined by the whole Christian Church, down to the time of the Reformation of the Calendar by Pope Gregory XIII.; and by which it might be determined *for ever*, had the length of the Julian year, and the assumed exact coincidence of 235 Lunations with 19 Julian years—the two essential elements on which the Old Church Calendar was based—been rigidly true. In using this Table it must be borne in mind that the Golden Numbers indicate, not (as in the General Table, p. 98) the days of the Paschal *New* Moons, but the days of the Paschal *Full* Moons, or fourteenth days, the latter being derived from the former by adding 13 clear days to the date of each New Moon; or, in other words, by moving down all the 19 Golden Numbers, beginning with XVI. (March 8), and ending with VIII. (April 5), 13 places each; so that, for example, XVI. may stand opposite March 21, and VIII. opposite April 18; the remaining Golden Numbers occupying the intermediate places, their *relative* position continuing unchanged.

THE OLD PASCHAL TABLE.

Golden Numbers.	Paschal Full Moons.	Calendar Letters.	Golden Numbers.	Paschal Full Moons.	Calendar Letters.
XVI.	March 21	C		April 8	G
V.	,, 22	D	XVII.	,, 9	A
	,, 23	E	VI.	,, 10	B
XIII.	,, 24	F		,, 11	C
II.	,, 25	G	XIV.	,, 12	D
	,, 26	A	III.	,, 13	E
X.	,, 27	B		,, 14	F
	,, 28	C	XI.	,, 15	G
XVIII.	,, 29	D		,, 16	A
VII.	,, 30	E	XIX.	,, 17	B
	,, 31	F	VIII.	,, 18	C
XV.	April 1	G		,, 19	D
IV.	,, 2	A		,, 20	E
	,, 3	B		,, 21	F
XII.	,, 4	C		,, 22	G
I.	,, 5	D		,, 23	A
	,, 6	E		,, 24	B
IX.	,, 7	F		,, 25	C

To find Easter by this Table, look for the Golden Number of the year in the first column, against which stands the day of the Paschal Full Moon. Then look in the third column for the Sunday Letter of the year next *after* the day of the Full Moon (Art. 76), and the day of the month standing against that Sunday Letter is *Easter Day*. If the Full Moon happen on a Sunday, then the next Sunday after is Easter Day.

The Golden Number may be found from the Tables, Art. 58; or from the Rule, "Add 1 to the year of Our Lord: divide the sum by 19. The remainder, if any, is the Golden Number. If there be no remainder, the Golden Number is 19 "(Art. 56).

The SundayLetter may be found by inspection from the Table, Art. 29; or by the Rule (Art. 33):—Add to the year of Our Lord its fourth part, omitting fractions. From the sum subtract the number 2; divide the result by 7. If there be no remainder, A is the Sunday Letter. But if any number remain, then the Letter corresponding to that number in the following scale is the Sunday Letter:—

A	G	F	E	D	C	B
0	1	2	3	4	5	6

In Bissextile, or Leap years, the Letter found as above will be the Sunday Letter and as Easter always falls after the intercalated day, this Letter will be the Easter Letter (Art. 27).

Ex. 1. Find Easter Day, A.D. 1350. We get by Table, Art. 58, G. N. 2, or by Rule $\left(\frac{x+1}{19}\right)_r = 2$; and by Table, Art. 29, we find Sunday Letter C; or by the Rule above given

$$\left(\frac{x + \left(\frac{x}{4}\right)_w - 2}{7}\right)_r = \left(\frac{1685}{7}\right)_r = 5 = \text{C}.$$

G. N. 2 stands against March 25; and the next C stands against March 28; which is, therefore, Easter Day.

Ex. 2. Find Easter Day in A. D. 1709, Old Style. Here we find G. N. XIX.; Sunday Letter B; both which stand against April 17: consequently, Easter Day is April 24.

61. The following Table exhibits another form in which the old Paschal Table was written, and which we find in the New Calendar published in England in 1561, and inserted into the Prayer Book of James (1604), the Scotch Liturgy of 1637, and the Caroline Prayer Book (1662). It was entitled a Table

To find Easter for Ever.

Golden Number.	A	B	C	D	E	F	G
I.	April 9	10	11	12	6	7	8
II.	March 26	27	28	29	30	31	April 1
III.	April 16	17	18	19	20	14	15
IV.	,, 9	3	4	5	6	7	8
V.	March 26	27	28	29	23	24	25
VI.	April 16	17	11	12	13	14	15
VII.	,, 2	3	4	5	6	March 31	April 1
VIII.	,, 23	24	25	19	20	21	22
IX.	,, 9	10	11	12	13	14	8
X.	,, 2	3	March 28	29	30	31	April 1
XI.	,, 16	17	18	19	20	21	22
XII.	,, 9	10	11	5	6	7	8
XIII.	March 26	27	28	29	30	31	25
XIV.	April 16	17	18	19	13	14	15
XV.	,, 2	3	4	5	6	7	8
XVI.	March 26	27	28	22	23	24	25
XVII.	April 16	10	11	12	13	14	15
XVIII.	,, 2	3	4	5	March 30	31	April 1
XIX.	,, 23	24	18	19	20	21	22

The Letters at the top are the Sunday Letters. The figures refer to the month on the same *horizontal* row with them.

To find Easter by this Table. The intersection of the vertical column, under the

Sunday Dominical Letter of the year, with the horizontal line containing the G. N. of the year, will give Easter Day: *e. g.*, let D and XVI. be the Sunday Letter and G. N. of the given year; then Easter Day falls on March 22.

This Table is, as I have said, evidently only another form of the old Paschal Table (Art. 60). As the Paschal Full Moons (or fourteenth days) may (each) fall on any day of the week, the Easter Day denoted by any G. N. may fall on any one of the seven days.

This Table, corrected for the change of style, was inserted in the Prayer Book at the same time as the other corrected Table from which it is derived. Like the latter, it will hold only until the year 1899, inclusive. This Table may be presented in another form, which is also useful in calculating Easter in the Julian Calendar.

Another Table to find Easter for Ever, according to the Unreformed Julian Calendar.

Almanac.	Dom. Lett.	Golden Numbers.					Easter.
1		XVI.					22 M.
8		V.	XIII.	II.	X.		29 M.
15	D	XVIII.	VII.	XV.	IV.	XII.	5 A.
22		I.	IX.	XVII.	VI.		12 A.
29		XIV.	III.	XI.	XIX.	VIII.	19 A.
2		XVI.	V.				23 M.
9		XIII.	II.	X.	XVIII.		30 M.
16	E	VII.	XV.	IV.	XII.	I.	6 A.
23		IX.	XVII.	VI.	XIV.		13 A.
30		III.	XI.	XIX.	VIII.		20 A.
3		XVI.	V.				24 M.
10		XIII.	II.	X.	XVIII.	VII.	31 M.
17	F	XV.	IV.	XII.	I.		7 A.
24		IX.	XVII.	VI.	XIV.	III.	14 A.
31		XI.	XIX.	VIII.			21 A.
4		XVI.	V.	XIII.			25 M.
11		II.	X.	XVIII.	VII.		1 A.
18	G	XV.	IV.	XII.	I.	IX.	8 A.
25		XVII.	VI.	XIV.	III.		15 A.
32		XI.	XIX.	VIII.			22 A.
5		XVI.	V.	XIII.	II.		26 M.
12		X.	XVIII.	VII.	XV.		2 A.
19	A	IV.	XII.	I.	IX.		9 A.
26		XVII.	VI.	XIV.	III.	XI.	16 A.
33		XIX.	VIII.				23 A.
6		XVI.	V.	XIII.	II.		27 M.
13		X.	XVIII.	VII.	XV.	IV.	3 A.
20	B	XII.	I.	IX.	XVII.		10 A.
27		VI.	XIV.	III.	XI.		17 A.
34		XIX.	VIII.				24 A.
7		XVI.	V.	XIII.	II.	X.	28 M.
14		XVIII.	VII.	XV.	IV.		4 A.
21	C	XII.	I.	IX.	XVII.	VI.	11 A.
28		XIV.	III.	XI.	XIX.		18 A.
35		VIII.					25 A.

This Table shows the possible combinations of each Sunday Letter with the 19 Golden Numbers. The total number of such combinations is 133 (7 × 19). The same is, of course, true of the last Table, p. 81. We shall see a similar Paschal Table in the Gregorian Calendar, where each Letter is combined with 31 Epacts, making the total number of combinations 271 (*infra*, Art. 134).

The Letter D is taken first, because it is the Sunday Letter for the earliest possible Easter Day, viz., March 22. The only Golden Number that will give this day is XVI. Similarly, the only Golden Number that will give the other extreme limit (April 25) is VIII., combined with Sunday Letter C. These are the two necessary and sufficient conditions; as XVI. and D were in the former extreme case.

The greatest number of Golden Numbers which, in combination with any Sunday Letter, can give the same day of the month for Easter is *five;* and the reason is that only 5 Golden Numbers, at most, can be included within the interval of 7 days.

To find Easter by this Table. Under the Sunday Letter of the year look for the G. N. of the year; and the corresponding date is Easter Day: *e.g.*, given Dom. Lett. D and G. N. XVII.; we find by inspection that Easter falls on April 12.

For explanation of column headed "Almanac," see Art. 165.

Concurrents, Regulars, and Keys of the Moveable Feasts.

62. Bede (*De Tempor. Rat.*, ch. 51; Ideler, ii. p. 261) defines the Concurrents to be the numbers which denote the day of the week on which the 24th of March falls, in the successive years of the Solar Cycle, in the scale.

1	2	3	4	5	6	7
Sun.	Mon.	Tu.	Wed.	Th.	F.	Sat.

Now, the Solar Cycle began the ninth year before A. D. 1 (*Vid.* Table, Art. 27), a Leap-year whose Sunday Letters were G F. Now, as the Calendar Letter of March 24 is F, it follows that in the first year of the Solar Cycle, March 24 fell on Sunday, and, therefore, its Concurrent was 1. The second year of the Solar Cycle, March 24 fell a day later, viz., on Monday, Concurrent 2: the third year, on Tuesday, Concurrent 3: the fourth year, on Wednesday, Concurrent 4: the fifth year (Leap-year), on Friday, Concurrent 6, and so on. They are called Concurrents because the numbers *run* on with the *course* of the Solar Cycle.

The following Table shows the relation existing between them and each year of the Solar Cycle, with the corresponding Sunday Letters: which in the Julian Calendar was repeated in the same order during every successive cycle. The asterisks denote the Leap-years.

TABLE I.

	Years of Sol. Cycle.	Concur.	Sunday Letter.	Years of Sol. Cycle.	Concur.	Sunday Letter.
B.C. 9.	* 1	1 Sun.	G F	15	4 W.	C
	2	2 M.	E	16	5 Th.	B
	3	3 Tu.	D	* 17	7 Sat.	A G
	4	4 W.	C	18	1 Sun.	F
	* 5	6 F.	B A	19	2 M.	E
	6	7 Sat.	G	20	3 Tu.	D
	7	1 Sun.	F	* 21	5 Th.	C B
	8	2 M.	E	22	6 F.	A
	* 9	4 W.	D C	23	7 Sat.	G
A.D. 1.	10	5 Th.	B	24	1 Sun.	F
	11	6 F.	A	* 25	3 Tu.	E D
	12	7 Sat.	G	26	4 W.	C
	*13	2 M.	F E	27	5 Th.	B
	14	3 Tu.	D	28	6 F.	A
				* 1	1 Sun.	G F
				&c.	&c.	

Here we see that the relation of the Concurrents to the Sunday Letter is—

Concurrent,	1	2	3	4	5	6	7
Sunday Letter,	F	E	D	C	B	A	G.

To find the Concurrent for any year A. D., we have first to find the number of the Solar Cycle corresponding to that year, and compare that number with the corresponding Concurrent in the above Table.

To find the Concurrent for any year A. D. (n), we have $\left(\frac{n+9}{28}\right)_r$ = the year of the Solar Cycle; then look out that year in the above Table, with its corresponding Concurrent: *e.g.*, Find the Concurrent for A. D. 532. We have $\left(\frac{532+9}{28}\right)_r = 9$, year of Solar Cycle, opposite to which we find 4 the Concurrent: therefore the 24th of March fell on *Wednesday* that year.

The Concurrents answer the same purpose as the Sunday Letters—viz., to find the day of the week corresponding to any given day of the year. They were an invention of the Easterns, as the Sunday Letters were of the Westerns; and with the REGULARS were employed for the finding of the week-day on which the Easter Full Moon fell, as I now proceed to point out.

The Calendar *New* Moons were indicated, as we have seen, by the *Golden* Numbers; and the fourteenth day, reckoned from New Moon inclusively, was the day of *Full Moon*.

Hence, if we desire to know the *day* of the *week* on which the Easter Full Moon (fourteenth day) fell in any year A. D., in the Old Calendar, we first find the Golden Number of the year—$\left(1 + \left(\frac{n}{19}\right)_r\right)$; and find the place of that Golden Number in the Paschal Table (Art. 60): we there find the number of the year in the Solar Cycle $\left(\left(\frac{n+9}{28}\right)_r\right)$; and from the Table, p. 84, the corresponding *Concurrent*, which gives the day of the week on which March 24 falls that year. Count 14 days from the day of New Moon (inclusive); and we get the day of month on which the Paschal *Full* Moon falls; and, knowing the day of week on which March 24 falls, we easily find the day of *week* on which the Full Moon falls. The *following* Sunday is *Easter Day*.

Reckon the number of days from March 24, exclusive, to the day of Full Moon, inclusive; divide that number by 7, and add the *remainder* to the Concurrent of the given year, and we get (by omitting multiples of 7) the number of the day of the week on which the Full Moon falls. If we divide the number of days elapsed between March 24 and the day of the Full Moon by 7, the remainder is called the REGULAR.

Ex. 1. Find the week-day of Easter Monday, A. D. 532.

$$\text{G. N.} = 1 + \left(\frac{532}{19}\right)_r = \text{I.},$$

which, in Table, Art. 60, corresponds to April 5. The number of days between March 24 and April 5 is 12, and the Regular is therefore 5.

The number of the year of the Solar Cycle is

$$\left(\frac{532+9}{28}\right)_r = 9,$$

and therefore, by Table I. of this Art., Concurrent is 4.

The number of the day of the Full Moon is therefore $5 + 4 = 2$ (after striking out the 7): *i. e.*, it falls on Monday.

Ex. 2. Week-day of Easter Full Moon, A. D. 425.

G. N. = VIII., and the Full Moon therefore falls on April 18, which is 25 days distant from March 24. The Regular is therefore 4.

The Concurrent is found to be 3, and $4 + 3 = 7$: that is, the Full Moon falls on Saturday.

Ex. 3. Week-day of Easter Full Moon, A. D. 813.

G. N. = XVI.: *i. e.*, Full Moon falls on March 21. This is 3 days *before* March 24: the Regular therefore is

$$\left(\frac{-3}{7}\right)_r = 4.$$

The Concurrent is 5, and 4 + 5 = 2 (leaving out the 7): *i. e.*, the Full Moon falls on Monday.

The general expression for the Regulars is easily found. Let P be the day of March (or of April, regarded as a continuation of March) on which the Paschal Full Moon falls, on every given year n; then, if R be the Regular,

$$R = \left(\frac{P - 24}{7}\right)_r. \qquad (1)$$

Let K denote the Concurrents of the given year; then the *week*-day on which P falls is given by this expression—

$$\text{Week-day of } P = \left(\frac{K + R}{7}\right)_r. \qquad (2)$$

As P depends on the *Golden Number* of the year, it is convenient to find the direct relation between the Golden Numbers and Regulars for any cycle of 19 years; which relation will hold good for all such cycles.

Now the inspection of the Paschal Table (Art. 60) shows that

For G. N.	I., P falls on	April	5	= March 36
„	II., „	„		= March 25
„	III., „	„	13	= March 44
„	IV., „	„	2	= March 33
„	V., „	„		= March 22,
	&c.	&c.		
„	XVI., „	„		= March 21.

Hence,

For G. N. I., $R = \left(\frac{36 - 24}{7}\right)_r = 5$

„ II., „ $= 25 - 24 = 1$

„ III., „ $= \left(\frac{44 - 24}{7}\right)_r = 6$

„ IV., „ $= \left(\frac{33 - 24}{7}\right)_r = 2$

„ V., „ $= 22 - 24 = 22 + 7 - 24 = 5$, adding 7,

&c.

„ XVI., „ $= 21 - 24 = 21 + 7 - 24 = 4$ „

&c.

Hence we have the following relation between the 19 Golden Numbers of any cycle and the corresponding *Regulars:*

TABLE II.

G. N.	I.,	II.,	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.
Regulars,	5,	1,	6,	2,	5,	3,	6,	4,	7,	3,	1,	4,	7,	5,	1,	4,	2,	5,	3.

Tables I. and II. are thus sufficient to find the *week*-day of P, and therefore the *month*-day of Easter Day, for any year n.

For, knowing the year of the Solar Cycle $= \left(\frac{n+9}{28}\right)_r$, we find, from Table I., the Concurrent, K; and knowing the G. N. $= 1 + \left(\frac{n}{19}\right)_r$, we find, from Table II., the *Regular*, R; and, consequently, the week-day of P, $\left(\frac{K+R}{7}\right)_r$.

CLAVES TERMINORUM.

Another term employed by the ancient Computists was the *Claves Terminorum*, or Keys of the *Moveable Feasts*.

These Keys were used in order to find the days of the month on which the five Moveable Feasts fell—viz., Septuagesima Sunday; the first Sunday in Lent; Easter Day; Rogation Sunday (fifth Sunday in Lent), and Pentecost.

The CLAVIS TERMINORUM for any year is the number which, when added to March 10, gives the date of the Easter Full Moon (Ideler, ii. 369); the nearest Sunday after is Easter Day.

Now Septuagesima and the first Sunday in Lent are respectively 9 and 6 weeks before Easter Day, and Rogation Sunday and Pentecost 5 and 7 weeks after Easter Day; and the same number of weeks, respectively, intervene between March 10, and January 6, January 27, April 14, and April 28; it follows that, if we add the Clavis of any year to these last four dates, we shall have the week-day immediately preceding Septuagesima, first Sunday in Lent, Rogation Sunday, and Pentecost.

Hence,

January 6	was called	the Term of	Septuagesima,
„ 27	„	„	first Sunday in Lent,
April 14	„	„	Rogation Sunday,
„ 28	„	„	Pentecost.

For example, the Clavis of A. D. 532 was 26.

Hence the Sundays which follow next after February 1, February 22, May 10, and May 24, are, respectively, the above-named Sundays.

To find the Clavis Terminorum for any year, we have to find the number of days between March 10 and the date of the Paschal Full Moon. Now we have already seen (p. 86) that, for

$$\begin{array}{lll} \text{G. N.} & \text{I.}, & P = \text{March } 36 \\ \quad " & \text{II.}, & " = \text{March } 25 \\ \quad " & \text{III.}, & " = \text{March } 44, \\ & \&\text{c.} & \quad \&\text{c.}; \end{array}$$

so that, if K denote the Clavis for any year A. D., and P the date of the Paschal Full Moon that year,

$$K = P - 10; \tag{3}$$

and, substituting the dates of P for the successive Golden Numbers, as given above (p. 86), we get for

$$\begin{array}{lll} \text{G. N.} & \text{I.}, & K = 36 - 10 = 26 \\ \quad " & \text{II.}, & " = 25 - 10 = 15 \\ \quad " & \text{III.}, & " = 44 - 10 = 34 \\ \quad " & \text{IV.}, & " = 33 - 10 = 23. \\ \quad " & \text{V.}, & " = 22 - 10 = 12, \\ & \&\text{c.} & \quad \&\text{c.}, \end{array}$$

so that we have the following relations between the Golden Numbers and the corresponding Claves:—

TABLE III.

G. N.	I.,	II.,	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.
Claves,	26,	15,	34,	23,	12,	31,	20,	39,	28,	17,	36,	25,	14,	33,	22,	11,	30,	19,	38.

Hence, to find the Clavis Terminorum for any year, we have only to find the Golden Number, and the corresponding number in Table III. is the required Clavis.

E. g., Find the Clavis for A. D. 532. Golden Number is I., and therefore Clavis is 26. As XVI. (March 21) is the first Golden Number within the Paschal limits, the *Minimum* Clavis is $21 - 10 = 11$; and as VIII. is the last Golden Number within the limits (April 18), the *maximum* Clavis is 39.

According to some writers, the terms of Septuagesima, first Sunday in Lent, Easter Day, Rogation Sunday, and Pentecost, are one day more than those above stated—viz., January 7, January 28, March 11, April 15, and April 29. But it is obvious that this merely makes the difference of a unit in the values of the Claves.

The above, respecting Concurrents, Regulars, and Claves, applies only to the Old Calendar, for which these auxiliary numbers were invented. But it is plain that they may also be applied to the Gregorian Calendar, by making use of the proper Paschal Table; if the Golden Numbers be used, the suitable Golden Number Paschal Table; or, if the Epacts be employed, using the corresponding Epacts.

63. We have now, by means of the foregoing Tables, completely solved the problem of finding the Old Style Easter. It can also be done *without the use of Tables* by means of a formula depending on the Golden Number and the Sunday Letter. The investigation of this problem is as follows.

First of all, we must substitute for the series of Golden Numbers another series of numbers called "Epacts," having a fixed relation to the former. To find these new numbers it is only necessary to recall the process by which the series of Golden Numbers was itself constructed (Art. 51). The "Epact," as here employed, "is the Calendar Moon's age at the end of any year, or, which is the same thing, at the beginning of the following year." Now—starting from G. N. III., affixed in the Calendar to Jan. 1, when the Moon's age was 0, and making the length of the Lunations 30 and 29 days, alternately—the Lunar year, consisting of 12 such months, will contain 354 days; and therefore, at the *end* of that Solar year (365 days) the Moon's age will be 11 days; in other words, the Moon's age at the beginning of the following year (G. N. IV.) will be 11. Similarly, at the *end* of this year, and at the beginning of the following year (G. N. V.), the Moon's age will be 22. At the end of year V. the Moon's age will be 33 days; an intercalary month of 30 days is inserted, and 3 days left to be carried on to year VI.; that is to say, the Epact of year VI. is 3. At the end of year VI., and the beginning of VII., the Moon's age will be 14 days. Proceeding, in the same way, throughout the successive years of the nineteen-year cycle,—making an intercalary month of 30 days, when the Moon's age exceeds 30, and carrying on to the following year the remainder, which becomes the Epact of that year,—we arrive at the following series of 19 *Epacts*, corresponding to the 19 Golden Numbers, each to each respectively :—

G. N.	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.,	I.,	II.
Epacts.	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	8	19.

The relation between each G. N. and its corresponding Epact may be thus expressed. —Let N be any G. N., and ϵ its corresponding Epact:

$$= \left(\frac{11\ (N-3)}{30}\right)_r = \left(\frac{11\ N-3}{30}\right)_r.$$

N

64. The following Paschal Table exhibits the relation between the Epacts and Golden Numbers, so as to enable us to see how the latter may at once be substituted for the former, in the finding of Easter Day.

Days.		Sun. Letter.	G. N.	Epact.	Days.		Sun. Letter.	G. N.	Epact.
	March.				April.	March.			
	21	C	XVI.	23	8	39	G		
	22	D	V.	22	9	40	A	XVII.	4
	23	E			10	41	B	VI.	3
	24	F	XIII.	20	11	42	C		
	25	G	II.	19	12	43	D	XIV.	1
	26	A			13	44	E	III.	0
	27	B	X.	17	14	45	F		
	28	C			15	46	G	XI.	28
	29	D	XVIII.	15	16	47	A		
	30	E	VII.	14	17	48	B	XIX.	26
April.	31	F			18	49	C	VIII.	25
1	32	G	XV.	12	19	50	D		
2	33	A	IV.	11	20	51	E		
3	34	B			21	52	F		
4	35	C	XII.	9	22	53	G		
5	36	D	I.	8	23	54	A		
6	37	E			24	55	B		
7	38	F	IX.	6	25	56	C		

The Golden Numbers in this Table indicate as before (p. 80) the Paschal Full Moons, or fourteenth days after New Moon (inclusive); the corresponding Epacts indicate the same. When the G. N. is given, the earliest day on which Easter can fall is the day after (15th), when that day is Sunday: for example, with the G. N. X., the earliest Easter is when the Sunday Letter is C, in which case Easter Day will fall on the day after the 27th, viz., the 28th of March.

Now, let N be the G. N. of any given year, and P the day of the month next after that to which the G. N. is affixed, that is to say, the earliest Easter with that G. N.; and let λ be the corresponding Calendar Letter; and let π be the day on which Easter actually falls, and L the Sunday Letter of the year; then, the equation for finding Easter is, obviously,

$$\pi = P + L - \lambda. \qquad (1)$$

For example, suppose the G. N. of the given year is X. By the Table we see that X. is affixed to March 27th; the day *after* which (15th day of the Moon) is March 28; the corresponding Calendar Letter is C. Suppose, further, that E is the Sunday Letter of the given year: then, as E is three places lower down than C, Easter Day will be three places lower down than P; that is to say, it will fall on March 30.

Returning now to the general expression above, we can always find L by the formula (4), (Art. 33); and P and λ can be deduced from the Epact (ϵ), which, as we have seen, is itself a function of the G. N. We have, then, to find the relation of P and λ, respectively, to ϵ.

Now, looking at the Table just given, we see that *for the earliest possible Easter* (March 22) $P = 22$; $\epsilon = 23$; and $L = \lambda = \mathrm{D} = 4$ (Art. 33 (4)). Hence, in this particular case

$$P + \epsilon = 45, \quad P = 45 - \epsilon; \tag{2}$$
$$\lambda + \epsilon = 27, \quad \lambda = 27 - \epsilon. \tag{3}$$

But it is easy to see that (2) and (3) are *general;* for, as P increases, ϵ diminishes by the same number; if P becomes $P + a$, ϵ simultaneously becomes $\epsilon - a$; consequently, the sum of P and ϵ is *always* constant. Similarly, the sum of λ and ϵ is also always constant; because when ϵ becomes $\epsilon - a$, λ becomes $\lambda + a$. Now, as Easter cannot be earlier than March 22, the minimum value of P is 22, when, by (2), the corresponding value of $\epsilon = 23$. If $\epsilon > 23$, we must instead of ϵ write its complement $30 - \epsilon$, so that,

$$(2) \text{ becomes } P = 45 + 30 - \epsilon = 75 - \epsilon; \tag{4}$$
$$\text{and } \lambda = 27 + 30 - \epsilon = 57 - \epsilon. \tag{5}$$

As λ denotes the *number* of a Calendar Letter, which cannot exceed 7, (3) and (5) must be written $\left(\frac{27 - \epsilon}{7}\right)_r$, $\left(\frac{57 - \epsilon}{7}\right)_r$; that is to say, λ will be the *remainder* after dividing (3) and (5) by 7.

In (1), $L - \lambda$ may be $= 0$; namely, when Easter Day falls on the 15th day of the Calendar Moon; but it cannot be *negative;* consequently, when $\lambda > L$, we must increase L by 7, and then subtract λ.

By means of the above equation (1) we can, without the use of any Tables, find Easter Day, when the Golden Number (N) and Sunday Letter (L) are given; or, as N and L are both functions of the *year number* (x), we can *find Easter Day from the single datum of the year number* (x). For,

$$N = \left(\frac{x+1}{19}\right)_r \text{ (Art. 56)}; \quad L = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w - 3}{7}\right)_r \text{ (Art. 33)};$$

$$\epsilon = \left(\frac{11 \times \left(\frac{x+1}{19}\right)_r - 3}{30}\right)_r \text{ (Art. 63)};$$

$$\left.\begin{array}{l} P = 45 - \epsilon \\ \lambda = \left(\dfrac{27 - \epsilon}{7}\right)_r \end{array}\right\} \text{ when } \epsilon \textit{ not} > 23;$$

$$\left.\begin{array}{l} P = 75 - \epsilon \\ \lambda = \left(\dfrac{57 - \epsilon}{7}\right)_r \end{array}\right\} \text{ when } \epsilon > 23.$$

Ex. 1. Find Easter Day for A. D. 1490.

Here $$N = \text{IX.}\,;\ \epsilon = 6\,;\ P = 45 - 6 = 39\,;\ L = C = 3\,;$$

$$\lambda = \left(\frac{27 - 6}{7}\right)_r = 0\,;\ L - \lambda = 3.$$

Hence $$\pi = P + L - \lambda = 39 + 3 = 42 \text{ of March}$$
$$= 11 \text{ of April.}$$

Ex. 2. Find Easter Day for A. D. 1207.

Here $$N = \text{XI}\,;\ \epsilon = 28\,;\ P = 75 - 28 = 47\,;\ \lambda = \left(\frac{57 - 28}{7}\right)_r = 1\,;$$

$$L = G = 7\,;\ L = \lambda = 6\,;$$

$$\therefore\ \pi = 47 + 6 = 53 \text{ of March}$$
$$= 22 \text{ of April.}$$

Ex. 3. Find Easter Day for A. D. 1424.

Here $$N = \text{XIX.}\,;\ \epsilon = 26\,;\ L = A = 1\,;$$

$$P = 75 - 26 = 49\,;\ \lambda = \left(\frac{57 - 26}{7}\right)_r = 3\,;\ L - \lambda = 1 - 3 = -2 = 5\,;$$

$$\therefore\ \pi = 49 + 5 = 54 \text{ of March}$$
$$= 23 \text{ of April.}$$

We shall see hereafter the modifications that these formulæ undergo in their application to the Gregorian Calendar (Art. 137).

65. As the New Moons of the *Lunar Cycle* recur on the same days of the month after the lapse of every period of 19 Julian years; and as the same days of the week recur on the same days of the month after the lapse of every 28 years (the *Solar Cycle*, Art. 26), it is obvious that by combining these two cycles, there results a cycle of 532 years (19 × 28), after the lapse of which the Calendar fourteenth Moons fall on the same days of the

month and week that they did 532 years before; and all the Moveable Feasts recur in exactly the same order as before (1). This famous Paschal *Cycle* is generally called by Chronologists the *Victorian Period* (2), from Victorius, Bishop of Aquetaine, who, at the request of Hilary, the Roman Deacon, drew up an Easter Canon founded on this cycle; which Canon was published by Hilary (after he became Pope) in the year 465. Victorius was not the inventor of this *Great Paschal Cycle*. An Egyptian monk, Arianus, had employed it, half a century earlier, in his *Chronographia*. He was, probably, the author of it; but this is not certain (3). This cycle has been erroneously attributed to Dionysius Exiguus, and called the *Dionysian Period;* and Scaliger further errs when he states (*De Cyclis Paschalibus*, p. 11) that Dionysius constructed an Easter Table of 532 years, which was continued by Bede from A. D. 1063 onward (4). What Dionysius *really* did was to adapt the 95-year cycle of Cyril of Alexandria to his new era of the Birth of Christ (5). However, when a Paschal Cycle of 532 years has once been drawn up (by means of the cycle of the Golden Numbers, and the cycle of the Dominical Letters), it may be extended backwards to all years B. C., and forwards to all years A. D., prior to the Reformation of the Calendar by Gregory XIII. To find the year of this cycle or period corresponding to any year A. D. or B. C., we must know some one of the epochs when it commenced; in other words, we must find a year in which the Golden Number was 1, and also the Solar Cycle 1. Now we know that A. D. 1 was 2 of the Golden Numbers, and 9 of the Solar Cycle: consequently, both a Solar and Lunar Cycle ended A. D. 75, and therefore so also did a Victorian Period (6). Hence A. D. 76 was the first year of a period, and so was 608, 1140, &c.; and in general, $76 + n \times 532$. To find what year of this period corresponds to any Julian year A. D., we have the following formula (where x denotes the given year, and V the corresponding year of the Victorian Period)—

$$V = \left(\frac{x - 75}{532}\right)_r; \text{ or, } = \left(\frac{x + 532 - 75}{532}\right)_r = \left(\frac{x + 457}{532}\right)_r.$$

If there be no remainder, x is the last year of a period; if there be a remainder, it will express the year of the period corresponding to x.

This period is employed in the useful work "On the Theory of the Calendar," by Francœur; and also by the late Professor De Morgan, in his "Book of Almanacs."

(1). As 19 and 28 are prime to each other, 532 (= 19 × 28) is the least common measure of both, and therefore the shortest period after which a complete identity can be produced.

(2). Vid. *Ideler*, ii., pp. 275, *sq.*

(3). *Ib.* p. 452.

(4). *Ib.* p. 292. The construction of this cycle from A. D. 532 to A. D. 1063 was the work of Bede himself, as plainly appears from his own words, cited by Ideler, *l. c.* If we will speak of a *Dionysian Period*, we must at least distinguish it from the Victorian, by assigning B. C. 1 as the epoch of the former, and B. C. 28 as that of the latter.

(5). In the *Epistola ad Petronium*, which formed the preface to his own Easter Table of 95 years, Dionysius expressly says, "Nonaginta quinque annorum hunc cyclum studio quo valuimus expedire contendimus."—Vid. *Ideler*, l. c., p. 292; and *Jan.* Hist. Cycl. Dion., p. 47.

(6). This may be arrived at thus. Let x be the number of years after A. D. 1, at which the cycle commences. Then, to determine x, we have the two conditions—

$$x + 1 = \text{mult. } (19) \qquad (1)$$
$$x + 9 = \text{mult. } (28). \qquad (2)$$

From (1) x must = some one of the series 18, 37, 56, 75 . . . .
From (2) x must = some one of the series 19, 47, 75 . . .

Hence 75 + 1 = 76 A. D. is the earliest date at which a Solar and a Lunar Cycle commence simultaneously.

66. If the Golden Numbers and Sunday Letters be determined for one period of 532 years, we can find them, as already said (Art. 65), for *any* year whatever, before or after that period, by finding the *corresponding* year of the period so determined. Thus, if the Golden Numbers and Sunday Letters for the first 532 years A. D. be ascertained, we can find the *Old Style* Golden Numbers and Sunday Letters for any year whatever A. D.; for A. D. 533 will have the same Golden Numbers and Sunday Letters as A. D. 1; 534 as 2; 535 as 3; and so on. To find the year of the period *corresponding* to any year (x) A. D., we have only to divide the year number by 532, and the *remainder* will be the corresponding year of the cycle. The general formula is

$$\left(\frac{x}{532}\right)_r.$$

Thus, for example, if we require the *Old* Style Golden Number and Sunday Letter for the year A. D. 3085, we have

$$\left(\frac{3085}{532}\right)_r = 425;$$

showing that A. D. 3085 has the same Golden Number and Sunday Letter as A. D. 425. The great advantage of this period is that it gives us *both* the elements necessary for the determination of Easter—namely, the Golden Number and Sunday Letter, *together*.

For the years B. C. there are, of course, no Easters, nor Sunday Letters, properly so called; but, by the Victorian Period, we can find, as above, the Golden Numbers and seventh-day Letters for any year B. C., by determining the *corresponding* year of the same period, as above. To find this, we have only to bear in mind that, since A. D. 1 corresponds to year 1 of that period, the year preceding A. D. 1—namely, B. C. 1—will be the last (532nd) of the preceding period; B. C. 2 will be the 531st; B. C. 3, the 530th; and in general, if V denote any year of any period,

$$V = 532 - (x - 1) = n \times 532 - (x - 1);$$

where n is to be taken so that $n \times 532$ may be the next multiple of 532 above $x - 1$.

Ex. 1. Find Golden Number and Sunday Letter for B. C. 151.

Here $n = 1$; and we have the year of the period—

$$V = 532 - 150 = 382.$$

Hence, the year 382 A. D. will give the required result.

Ex. 2. Find Golden Number and Sunday Letter for B. C. 1096.

Here $$n = 3, \text{ and } V = 1696 - 1095 = 501.$$

Hence the year 501 A. D. will give the required result (1).

(1). De Morgan, "The Book of Almanacs," London, 1851, gives a complete Victorian Period of 532 years, from A. D. 1 to 532, which repeats itself for ever in the Julian Calendar; each successive period commencing with A. D. 533, 1065, 1597, &c., respectively.

67. We have now considered the three great cycles employed in the construction of the Church Calendar—viz., the Solar Cycle, the Lunar Cycle, and the Victorian (Dionysian) Cycle, or Period, which is a compound of the other two. It remains to say a few words respecting a fourth cycle, extensively used in chronological dates, from the fourth century onwards through the Middle Ages. I mean the *Cycle of the Indiction* (1). This is a period of 15 years, reckoned consecutively as Indiction 1, Indiction 2, &c., up to 15; after which the cycle recurs in the same order. The first express mention of it as a chronological date(2) is in an edict of Constantius, A. D. 356 or 357. By the author of the Paschal Chronicle, its origin is assigned to the time of Constantine, and its *epoch* fixed on September 1, A. D. 312. Dionysius Exiguus followed him (3), and gave the Rule for finding the Indiction corresponding to any year A. D., which has been repeated in all works on Chronology:—viz., "Add 3 to the year A. D., and divide the sum by 15: if there be no remainder, the Indiction for that year is 15; if there be a remainder, that remainder is the number of the year of the cycle." Hence,

$$\left(\frac{x + 3}{15}\right)_r$$

is the general formula, x being the given year.

According to this Rule, a cycle *ended* A. D. 312; because 312 + 3 = 315, which, divided by 15, leaves no remainder.

But this apparent contradiction is removed when we remember that, while a new cycle *began* on the 1st of September that year, the preceding cycle *ended* on the 31st of August; and the same is true of all the Indiction Cycles—one cycle *ends* and the other *begins* in the same year. The Rule above gives the Indiction which belongs to the *greater* part (the first eight months) of a year: if we require the Indiction which *begins* in a year, and to which the last four months belong, we must add 4, instead of 3, to the year number; that is to say, the formula is

$$\left(\frac{x+4}{15}\right)_r.$$

—*Vid.* Ideler, ii. 357.

Thus, for the year 1875, the Indiction given by the Rule is 3—

$$\left(\frac{1875+3}{15}\right)_r;$$

but that Indiction ends on the 31st of August, and a new one begins on September 1(4).

(1). Vid. *Sir H. Nicolas*, Chron. of Hist., p. 6; *Ideler*, i. p. 72; ii. pp. 347–364. The probable cause why the cycle contained 15 years was that a provincial *census*, for taxation, took place after an interval of three Roman Lustra.

(2). *Ideler*, ii. 352.

(3). *Ib.* 356. Others, and among them the authors of "L'Art de Verifier les Dates," give 313 as the Epoch.

(4). Besides the Indiction Cycle described in the text, there are usually reckoned *three* others, also of 15 years each, but differing as to the day of their commencement. The *Imperial*, or Cæsarean, began on September 24; the Roman, or *Pontifical*, on January 1; and the *French* began in October.—Vid. *Nicolas*, l. c., p. 7; and *Ideler*, l. c., pp. 359-364.

68. If we multiply together the three cyclical numbers 28, 19, and 15 (the Solar, Lunar, and Indiction Cycles), we get 7980. A period of 7980 years will, therefore, bring round the years of the three cycles again in the same order; so that each year of the compound period will hold the same place in each of the three component cycles as the corresponding year in the preceding period. And as the above three cyclical numbers are prime to each other, it follows that no two years in the compound period can hold the same place in *all* the three component cycles. Hence, to specify the numbers of a year in each of these cycles is, in fact, to identify the year in the period; provided the year be contained within the period which embraces the whole of authentic Chronology.

Vice versâ, if any year of the period be given, we can find the corresponding cyclical numbers, which are the remainders arising from dividing that year by 28, 19, and 15, respectively. For example, the year 6539 of the period has 15, 3, 14, for its numbers in the Solar, Lunar, and Indiction Cycles, respectively.

This period of 7980 years was called by its inventor, Joseph Scaliger, the JULIAN PERIOD, because it reckons by Julian years [1]. Scaliger is supposed by some to have learned it from the Greeks of Constantinople. However this may be, its utility is so great that it has been employed by all Chronologers since his time: and Ideler, a most competent authority, says (i. 77) of it, that by means of it light and order were first introduced into Chronology. The period contains 285 Solar Cycles, 420 Lunar, and 532 Indictions.

(1). *Vid.* J. Scaliger, Emend. Tempor., lib. v., p. 359; edit. 1629.

69. The answer to the question "Given the three cyclical numbers belonging to any year of the period, to find that year," depends on the solution of a problem in indeterminate analysis. But instead of directly solving this problem, it is preferable to determine what year of the Julian Period corresponds to the first year of the Christian Era; because, when this has been done, it is easy to determine the relation between all the other years of the Julian Period, and the corresponding years of the Christian reckoning, and thus to solve indirectly the problem above stated.

To find the year of the Julian Period corresponding to the first year of the Christian Era, we have the following data:—In A. D. 1 the number of the Solar Cycle was 10, of the Lunar Cycle 2, and the Indiction 4: and the problem is to find a number such that, when divided by 28, 19, and 15, the remainders shall be 10, 2, and 4, respectively. The required number is 4714 [1]. Hence, if n be any year A. D., and x its equivalent in the Julian Period,

$$x = \left(\frac{n + 4713}{7980}\right)_r.$$

To reduce any year B. C. (n') to its equivalent year in the Julian Period (x), subtract it from 4714, or $x = 4714 - n'$: *e.g.*, B. C. 5 = 4709 of Julian Period [2]. If $4714 - n'$ be negative, we must add such a multiple of 7980 as will make it positive.

To reduce a Julian Period to the Christian Era, we have (if we know that the year lies between 4714 B. C. and 3267 A. D.) the Rule :—

Subtract the year of the Julian Period from 4714, if it be less; the result gives years B. C.: subtract 4713 from the year of the Julian Period, if it be greater; the result gives years A. D. Thus, 4709 gives B. C. 5; 4743 gives A. D. 30.

(1). The following is Ideler's solution of the problem (ii. 587):—

Let P be the year of the Julian Period which will coincide with A. D. 1; then, by the conditions of the question, P, when divided by 28, 19, and 15, successively, gives remainders 10, 2, and 4.

(1). Hence $\frac{x-10}{28}$ must be a whole number—call it A: then $x = 28\,A + 10$.

(2).
$$\frac{x-2}{19}, \text{ or } \frac{28\,A+8}{19} = A + \frac{9\,A+8}{19} = \text{an integer},$$

and, therefore,
$$\frac{9\,A+8}{19} = \text{an integer}, \text{ —call it } B: \text{ therefore}$$

$$A = \frac{19\,B-8}{9} = 2\,B + \frac{B-8}{9}:$$

therefore $\frac{B-8}{9}$ is also a whole number—call it C: then $B = 9\,C + 8$, whence, by substitution,

$$A = 19\,C + 16 \text{ and } x = 532\,C + 458.$$

(3). But
$$\frac{x-4}{15}, \text{ or } \frac{532\,C+454}{15} = 35\,C + 30 + \frac{7C+4}{15} = \text{an integer};$$

therefore, so also is
$$\frac{7\,C+4}{15} = \text{an integer} = D, \text{ say}.$$

Then
$$C = 2\,D + \frac{D-4}{7};$$

therefore $\frac{D-4}{7}$ = an integer $=E$, say: then $D = 7\,E + 4$, and $\therefore\ C = 15\,E + 8$, and hence, finally,

$$x = 7980\,E + 4714.$$

E may have any integer value, including 0: therefore the least value x can have is 4714: i. e., A. D. 1 *corresponds to the 4714th year of the first* (unexpired) *Julian Period.—Vid.* also *Delambre*, Astr. Theor. et Prat.; and Encycl. Brittan., Art. "Calendar."

The following is the Rule given by Beveridge (Instit. Chron.) to find the year of the Julian Period, when the year of the Solar Cycle, the Golden Number, and the Indiction, are given:—"Multiply the year of the Solar Cycle by 4845; the Golden Number by 4200; and the Indiction by 6916: add, and divide the sum by 7980, and the *remainder* is the required year of the Julian Period."

To prove this Rule, call n the number of the year A. D.; a, b, c the numbers of the Solar and Lunar Cycles, and the Indiction, respectively: then—

$$n = \text{mult. } 28 + a - 9,$$

or
$$a = n + 9 - \text{mult. } (28).$$

Similarly,
$$b = n + 1 - \text{mult. } (19),$$
$$c = n + 3 - \text{mult. } (15);$$

$$\therefore\ \left(\frac{4845\,a + 4200\,b + 6916\,c}{28 \times 19 \times 15}\right)_r = \left(\frac{15961\,n + 68553}{7930}\right)_r = \left(\frac{n+4713}{7980}\right)_r$$

= number of year of Julian Period, by this Article.

(2). With respect to the numbering of years B. C., the practice (as before observed) is not uniform. Chronologists reckon the year preceding A. D. 1 as the first of the backward (or B. C.) series. In this case $x = 4714 - n'$, as I have taken it. Astronomers, on the other hand, in order to keep up the continuity, mark with 0 the year before A. D. 1 (*vid.* Art. 21): and consequently, with them $x = 4713 - n'$.

70. It has already been observed (Art. 59), that the earliest and latest dates of Easter Day are, respectively, March 22 and April 25. The following Table shows the different years, from the commencement of the Christian Era to the date of the Reformation of the Calendar (1582), in which Easter Day has fallen on these extreme limits.

March 22, A. D.	Interval.	April 25, A. D.	Interval.
72		45	
	247 years.		95 years.
319		140	
	95 ,,		247 ,,
414		387	
	95 ,,		95 ,,
509		482	
	95 ,,		95 ,,
604		577	
	247 ,,		95 ,,
851		672	
	95 ,,		247 ,,
946		919	
	95 ,,		95 ,,
1041		1014	
	95 ,,		95 ,,
1136		1109	
	247 ,,		95 ,,
1383		1204	
	95 ,,		247 ,,
1478		1451	
	95 ,,		95 ,,
1573		1546	

Inspection of the above Table shows that, from the commencement of the Christian Era down to the year 1582, Easter Day has fallen on the 22nd of March only twelve times, and on 25th April also only twelve times. It appears further, that in the 1st, 4th, 5th, 6th, 7th, 10th, 11th, 12th, 15th, and 16th centuries, *both* occurred; while in the 3rd and 8th *neither* took place. It results, moreover, that the intervals of occurrence, in each case, are the same (though differently distributed), viz., 95 and 247 years, which are multiples of the Lunar Cycle: 19×5 and 19×13. The longer intervals (247 years) are separated from each other by three intervals of 95 years each. This is equivalent to saying that the Victorian Period of 532 years falls into four lesser periods—one of 247 years, the other three of 95 years each; in the last year of each of which four periods Easter Day falls on one of its extreme limits. For instance, in the 532 years from A.D. 72 to A.D. 604, Easter Day occurs in both these years on March 22nd. But it also falls on that day three times during the whole interval, viz., A.D. 319 (interval of 247 years), 414, and 509 (intervals of 95 years), the latter date being itself removed 95 years from A.D. 604. We shall see hereafter how this law, respecting the extreme Paschal limits, is modified in the Gregorian Calendar.

71. The reason of this law of succession may be easily seen. And first for March 22: the conditions of this case are two, viz.: 1st, the G. N. of the year must be XVI., and 2nd, the Sunday Letter must be D. Now, from the expression for the G. N. of any year x, viz. $\left(\frac{x+1}{19}\right)_r$, it is plain that A.D. 15 is the first year whose G. N. is XVI. The next is A.D. 34; the next 53; the next 72. But none of these except 72 fulfils the other condition of having the Sunday Letter D: consequently A.D. 72 is the first year in which Easter fell on March 22. It was Leap-year, its Letters being E D, the latter being the Easter Day Letter.

To find in what years Easter fell on March 22, we have the conditions—

$$x = 19\,m + 72, \qquad (1)$$

where m is an integer; also $\left(\frac{x + \left(\frac{x}{4}\right)_w - 1}{7}\right)_r = 5$, since 5 corresponds to Sunday Letter D; or,

$$x + \left(\frac{x}{4}\right)_w = \text{mult. } (7) + 6. \qquad (2)$$

Substituting from (1) in (2), we get

$$19\,m + 72 + \left(\frac{19\,m}{4}\right)_w + 18 = \text{mult. } (7) + 6,$$

or,

$$\left(\frac{95\,m}{4}\right)_w = \text{mult. } (7). \qquad (3)$$

This last condition gives us $95\,m = \text{mult. } (28) + r$, where r must be either 0, 1, 2, or 3.

Hence, $$\frac{95\,m - r}{28} = \text{integer}, \;\therefore\; \frac{11\,m - r}{28} = \text{integer} = A, \text{ say};$$

thus,

$$\frac{28A+r}{11} = \text{integer}, \therefore \text{ so also is } \frac{6A+r}{11} = B;$$

thus,

$$\frac{11B-r}{6} = \text{integer}, \therefore \text{ so also is } \frac{5B-r}{6} = C;$$

then,

$$\frac{6C+r}{5} = \text{integer}, \therefore \frac{C+r}{5} = \text{integer} = D.$$

Thus,

$$\begin{aligned} C &= 5D - r \\ B &= 6D - r \\ A &= 11D - 2r \\ m &= 28D - 5r, \end{aligned}$$

where D may have any integral value, so as to make m positive.

If we put $D = 1$, and r successively 0, 1, 2, 3, we get

$$m = 28, 23, 18, 13.$$

Putting $D = 2$, and r as before,

$$m = 56, 51, 46, 41.$$

The corresponding values of x are, by equation (1), giving them in the natural order,

For $D = 1$,
$$\begin{aligned} x &= 72 + 247 = 319 \\ x &= 72 + 247 + 95 = 414 \\ x &= 414 + 95 = 509 \\ x &= 509 + 95 = 604, \end{aligned}$$

the last completing the Victorian Cycle of 532 years.

For $D = 2$, $x = 851, 946, 1041, 1136,$
successively, and so on for higher values of D.

Thus the four smaller periods into which the whole period of 532 is divisible follow this law:—Reckoning from the beginning of a whole period, A. D. 72, the first sub-period occurs after 13 × 19 (= 247) years; the second, after 18 × 19 (= 342), *i. e.*, 95 years after the first; and similarly, the other two sub-periods follow at intervals of 95 years each (1).

In the same way, we can prove the law of the succession of Easter Days falling on April 25—the other extreme limit. The condition for this is G. N. VIII. and Sunday Letter C (Art. 59). Now, the first year of G. N. VIII. is A. D. 7; but the Sunday Letter for this year is not C but B: and in fact the first year which satisfies both conditions is A. D. 45 $(7 + 2 \times 19)$.

Hence, for our first condition we have

$$x = 45 + 19\,m, \tag{4}$$

m being integral; and as C corresponds to 6 in the scale, the second condition is

$$x + \left(\frac{x}{4}\right)_w = \text{mult. } (7) + 7 = \text{mult. } (7).$$

By substitution

$$19\,m + 45 + \left(\frac{19\,m + 1}{4}\right)_w + 11 = \text{mult. } (7),$$

or

$$\left(\frac{95\,m + 1}{4}\right)_w = \text{mult. } (7).$$

Proceeding as before

$$95\,m + 1 = \text{mult. } (28) + r,$$

where r is 0, 1, 2, 3, in succession.

So

$$\frac{11\,m + 1 - r}{28} = \text{integer} = A,$$

$$\therefore \frac{6\,A + r - 1}{11} = \text{integer} = B,$$

and

$$\frac{5\,B - \overline{r - 1}}{6} = \text{integer} = C,$$

$$\therefore \frac{C + \overline{r - 1}}{5} = \text{integer} = D.$$

Hence,

$$C = 5\,D - \overline{r - 1};\ B = 6\,D - \overline{r - 1};\ A = 11\,D - 2\,\overline{r - 1};$$

and finally,

$$m = 28\,D - 5\,\overline{r - 1} = 5 + 28\,D - 5r.$$

For $D = 0,\quad m = 5, 0;$
$D = 1,\quad m = 33, 28, 23, 18;$
$D = 2,\quad m = 61, 56, 51, 46;$

and the whole series of values of m are

0, 5, 18, 23, 28, 33, 46, 51, &c.

The corresponding values for x, from equation (4), are

45, 140, 387, 482, 577, &c.

The Victorian Period, reckoned from 45, therefore, may be divided into four smaller periods, which follow one another at the following intervals, beginning with 45,

95, 247, 95, 95.

We shall see hereafter (Art. 147) how the above law of succession is modified in the Gregorian Calendar.

(1). We might solve the problem quite generally, thus:—

Let x be the year A. D. in which the G N. XVI. and Sunday Letter D concur: then to find x we have

$$\left(\frac{x+1}{19}\right)_r = 16,$$

or

$$x = 19\,m + 15, \text{ where } m \text{ is an integer.}$$

Also

$$\left(\frac{x + \left(\frac{x}{4}\right)_w - 1}{7}\right)_r = 5, \text{ or } x + \left(\frac{x}{4}\right)_w = \text{mult. } (7) + 6.$$

By substitution, $19\,m + 15 + \left(\frac{19\,m+3}{4}\right)_w + 3 = \text{mult. } (7) + 6;$

or,

$$\left(\frac{95\,m+3}{4}\right)_w = \text{mult. } (7) - 12.$$

Proceeding as before, we get

$$95\,m + 3 = 4\,\{\text{mult. } (7) - 12\} + r,$$

where r must be 0, 1, 2, or 3. That is

$$95\,m + 51 - r = \text{mult. } (28);$$

or,

$$\frac{95\,m + 51 - r}{28} = \text{integer}, \therefore \frac{11\,m + 23 - r}{28} = \text{integer} = A, \text{ say};$$

$$\therefore \frac{28\,A - 23 + r}{11} = \text{integer, and } \frac{6\,A + r - 1}{11} = \text{integer} = B.$$

So

$$\frac{11\,B - r + 1}{6} = \text{integer, and } \frac{5\,B - r + 1}{6} = \text{integer} = C.$$

Thus

$$\frac{C + r - 1}{5} = \text{integer} = D, \text{ say}:$$

and therefore

$$\begin{aligned} C &= 5\,D - r + 1 \\ B &= 6\,D - r + 1 \\ A &= 11\,D - 2\,(r-1) \\ m &= 28\,D - 5\,r + 3. \end{aligned}$$

For $D = 0$, $r = 0$ gives $m = 3$.

$D = 1$, $r = 0, 1, 2, 3$, gives $m = 31, 26, 21, 16$.

$D = 2$, $r = 0, 1, 2, 3$, gives $m = 59, 54, 49, 44$.

By substituting in equation (1) of this Note, we get for x the successive values

72, 319, 414, 509, 604, &c., as before.

72. During the eight centuries between the time of Charlemagne and the close of the sixteenth century, the Alexandrian Easter Canon, which in Western Europe was called the Dionysian Canon, was universally accepted and followed, and no difference existed regarding the time of celebrating Easter. A new dispute, however, then arose on this subject, which has never since been entirely settled.

The Dionysian Easter Canon rests, as we have seen, on two suppositions, neither of which is strictly accurate; viz., 1°, that the Tropical year consists of 365¼ days exactly; and 2°, that 235 Synodic Lunations are exactly equal to 19 such (Julian) years. Now, the exact length of the Tropical year is (Art. 11) $365^d\ 5^h\ 48^m\ 47{\cdot}46^s$, that is, $11^m\ 12{\cdot}54^s$ less than the Julian year; while 235 Synodic Lunations are $1^h\ 28^m\ 46{\cdot}33^s$ shorter than 19 Julian years ([1]). The former error (that of the Julian year) would accumulate to 1 day in about 128⅓ years; or to 7 days in about 900 years. The latter error (that of the Lunar Cycle) would amount to a day in 308 years. Clavius, and the other mathematicians who corrected the Calendar for Pope Gregory, assumed the former error to amount to 1 day in 134 years, or 3 days in 400 years; and the latter error they made to be 1 day in 312½ years, or 8 days in 2500 years ([2]). The effect of these errors was that the Vernal Equinox and the mean New Moons gradually fell earlier and earlier in the year than they should have done had the Julian year been exact; the Equinox, at the rate of about a day every 128 years; the New Moons, about a day every 308 years. An obvious consequence of this was that neither the Immoveable nor Moveable Festivals remained in the places originally assigned to them in the Julian year. The former, being fixed to a stated day of the month, were occurring later and later in the actual Tropical year; and the Moveable Festivals, depending as they did on the Julian Vernal Equinox (March 21), were also falling later and later in the actual year. In other words, as the Vernal Equinox was fixed in the early Church to the 21st of March ([3]), about which time it then actually fell (Art. 41, Note 6), the effect of making the year too long was that the *nominal* 21st of March was every year falling behind, or *later* than, the actual Equinox; or, which is the same thing, the real Equinox was occurring *sooner* than March 21. In process of time, the seasons of the year, which are reckoned with reference to the real Equinox, were shifting *backwards;* and in the course of about 47,000 ($365 \times 128 = 46{,}720$) years would have *retrograded* through the whole year. The Immoveable Festivals, determined by the Julian Calendar, would always be falling later and later in the actual year, so that, *e. g.*, the Annunciation (25th March) would, in the course of time, take place at Midsummer. Similarly, Easter, which by the ancient rule was connected with the Julian 21st of March (Art. 41), would gradually fall more and more behind (or later than) the true Equinox; and in the course of time, it and all its dependent Festivals would fall successively in every season of the year, and indeed on every day of the year.

At the time of the Reformation of the Calendar by Gregory XIII., the error arising from the Julian year being too long amounted to very nearly 10 days; so that the true Equinox had *receded* nearly 10 days (4) from the nominal 21st of March. And the error of the Lunar Cycle amounted to more than 4 days (5), so that the fourteenth day of the *Calendar* Moon, as shown by the G. N., was the 18th of the *mean* Moon. Easter was no longer celebrated (as it ought to have been) from the 15th day of the Moon to the 21st, inclusive, but from the 19th to the 25th, inclusive. So that, in order to find the place of the mean New Moon it was usual to ascend (or go back) from the place of the current G. N. to the fifth day *inclusive;* which was done by the following five syllables, *in cœlis est hic:* or these, *hic nova luna;* or these, *in cœlo luna.*

For example, suppose the G. N. of the year was IX., which was affixed to March 25, the place of the mean New Moon was the 21st (the fifth day higher up):

Hic	no -	va	Lu -	na.
21	22	23	24	25.

Had no correction of this source of error taken place, it is obvious that in the course of time Easter Day would have been celebrated about the time of actual *New* (instead of Full) Moon (6).

(1).

(*a*)

$$\begin{aligned} \textit{Julian year} &= 365{\cdot}25^{d} &&= 365^{d}\ 6^{h} \\ \textit{Tropical year} &= 365{\cdot}242216^{d} &&= 365^{d}\ 5^{h}\ 48^{m}\ 47{\cdot}46^{s} \\ \text{Diff.,} &= \quad {\cdot}007784 &&= 0^{d}\ 0^{h}\ 11^{m}\ 12{\cdot}54^{s} \end{aligned}$$

(*b*) $\textit{Mean Astronomical Lunation} = 29{\cdot}5305887^{d} = 29^{d}\ 12^{h}\ 44^{m}\ 2{\cdot}87^{s}.$

$$\begin{aligned} 19\ \textit{Julian years} &= \quad 6939{\cdot}75^{d} = 6939^{d}\ 18^{h} \\ 235\ \textit{Mean Lunations} &= 235 \times 29{\cdot}530588^{d} = 6939^{d}\ 16^{h}\ 31^{m}\ 13{\cdot}27^{s} \\ \text{Diff.,} &= \qquad 1^{h}\ 28^{m}\ 46{\cdot}73^{s} \end{aligned}$$

$\textit{Mean Calendar Lunation} = \dfrac{6939\frac{3}{4}}{235} = 29^{d}\ 12^{h}\ 44^{m}\ 25{\cdot}5^{s}$, differing from Mean Astronomical Lunation by about 22^{s}.

(2). *Clavius* (p. 74), following the Alphonsine Tables, made the length of the Tropical year to be $365^{d}\ 5^{h}\ 49^{m}\ 46^{s}$: *i. e.*, $10^{m}\ 44^{s}$ shorter than the Julian year; which would amount to a day in about 134 years; or 3 days in about 400 years. The Synodic month he took (p. 102) to be $29^{d}\ 12^{h}\ 44^{m}\ 3{\cdot}10^{s}$.

(3). Sosigenes, who effected (B. C. 45) the Julian Reformation of the Calendar, assigned the Vernal Equinox to the 25th of March. It actually took place, then, according to Delambre's Tables, on the 23rd. Supposing Sosigenes to have been right, and allowing 1 day in 129 years for the regression of the actual from the nominal Equinox, the actual Equinox would, by calculation, fall on the 22nd of March, at the time of the Nicene Council (45 B.C. + 325 A. D. = 369; which, divided by 129, gives nearly 3). It actually fell on the 20th. The Computists reckoned it as falling on the 21st, and it was determined that this day should always be reckoned the day of the Vernal Equinox.

(4). 1582 – 325 = 1257 years. Hence, 1257 × ·007784 (Note 1, above) = 9·7945 days. By Delambre's Tables the Vernal Equinox in 1582 fell on March 11, 1 o'clock, A. M.

(5). $308^y : 1^d :: 1257^y : 4^d$ (nearly).

(6). This would take place as soon as the mean New Moon should have receded 15 days from the Calendar Moon, shown by the Golden Number, which would happen about A. D. 5000. For, as the recession, reckoned from the date of the Council of Nicæa (A. D. 325), is at the rate of 1 day in 308 years, we have the proportion $1^d : 308^y :: 15^d : 4620^y$. To which adding 325, we get 4945 A. D.

73. Attention was turned to this matter, in the West, as early as the beginning of the thirteenth century (*vid.* Art. 44) (1). In the East, also, towards the end of the fourteenth century (1372), Isaac Argyrus, a Greek monk, wrote a tract on the subject (2). In the fifteenth century the subject was brought before the Council of Constance (1414) by Cardinal P. d'Ailly, and before the Council of Basle (1436) by Cardinal Cusames. They were the first who suggested the omission of certain days, in order to restore the Equinox to the 21st of March. Towards the end of this century (1475), Pope Sixtus IV. was most desirous to effect the Reformation of the Calendar, and summoned the famous astronomer, Johannes Regiomantus, to Rome for that purpose. But the latter died the following year, before the undertaking was sufficiently matured. It was again mooted in the Lateran Council under Pope Leo X. (3); but more urgent matters interfered with it. At last, the Council of Trent formally delegated the task of reforming the Calendar to the Pope; and Gregory XIII., who attended this Council as a jurist, carried out this reformation in the year 1582. Among the various proposals submitted to him (4), he approved most of that the author of which was Luigi (Aloysius) Lilio (5), a native of Calabria. A compendium of this work of Lilio's was sent round by the Pope, in 1577, to all the Christian princes and the most famous universities, in order to obtain their opinions respecting it (6). A general approbation having been expressed, a commission was then appointed by the Pope to carry out the work. The principal member of this commission was a German Jesuit of Bamberg, a distinguished mathematician, named Christopher Schlüssel, better known by his Latinized name *Clavius*. To him the Gregorian Calendar in its present form is mainly due. The commission drew up, on the basis of Lilio's plan, and with a few changes in it, certain "*Canones in Calendarium Gregorianum perpetuum;*" and thereupon the Pope published a Bull founded on these Canons, dated February 24, 1581, enjoining the use of the Reformed Calendar (7). Clavius' great work, already so often referred to (*Romani Calendarii a Greg. XIII., Pont. Max., restituti explicatio, Clementis VIII. jussu edita*), in which the whole subject is developed and exhausted, did not appear till twenty-one years after (1603).

In the Bull just referred to, the reformation effected by Gregory's direction is thus described:—"*Considerantes nos ad rectam Paschalis Festi celebrationem tria necessario conjungenda et statuenda esse, primum, certam Verni Æquinoctii sedem; deinde rectam positionem* xiv. *Lunæ primi mensis, quæ vel in ipsum Æquinoctii diem incidit, vel ei proxime succedit; postremo, primum quemque diem Dominicum qui eandem* xiv. *Lunam sequitur, curavimus non solum Æquinoctium Vernum in pristinam sedem, a qua jam a Concilio Nicæno decem circiter diebus recessit, restituendum, et* xiv. *Paschalem suo in loco, a quo quatuor et eo amplius dies hoc tempore distat, reponendam; sed viam quoque tradendam et rationem, qua caveatur ut in posterum Æquinoctium et* xiv. *Lunæ a propriis sedibus nunquam dimoveantur.*"

(1). The first work on the subject was anonymous; the author was generally supposed to be Vincentius Bellovacensis. Herzog, Encycl., vol. vii. p. 230.—Vid. *Clavius*, Prœmium.

(2). This essay is to be found in *Petavius*' Uranologium, pp. 204, *sq.*

(3). *Paulus Middelburg.* addressed an urgent appeal to the Council on the subject, in which he specially advised the correction of the "*Numerus aureus, qui diuturnitate temporis jam factus est plumbeus.*" (*Ideler*, ii. p. 300).

(4). Vid. *Clavius*, Prœmium.

(5). Vid. *Ideler*, ii. 301. It is usual to speak of the *brothers* Luigi and Antonio Lilio as the authors of this plan. But Antonio, who was a physician in Rome, had no part in the work, except that he presented his brother's tract to the Pope. In the Pope's Bull we find, "Dum itaque nos in hac cogitatione curaque versaremur, allatus est nobis liber a dilecto filio, Antonio Lilio, artium et medicinæ doctore, quem quondam Aloysius, ejus germanus frater, conscripserat."

(6). This abstract of Lilio's book is printed by Clavius at the beginning of his own great work (pp. 3, *sq.*).

(7). This Bull is given at length by Clavius, *l. c.*, p. 13.

74. Before proceeding to show how the three great objects indicated in the Papal Bull just quoted were attained, it may be useful to dwell, a little more fully than has been done in Art. 72, on the actual state of the Church Calendar at that time, and the inconsistencies and perplexities in practice occasioned by its errors. It constantly happened that Easter Day was kept in direct violation of the Canons laid down by the Ancient Church (Art. 41) for its celebration. The *first month* (in which Easter must be kept) was defined to be that month whose fourteenth Moon falls on the day of the (*actual*) Vernal Equinox, or follows next after it. Now, as we have seen (Art. 72), the Vernal Equinox *actually* took place, in the sixteenth century, on the 11th of March. Hence it follows that all the fourteenth Moons, from the 11th of March (inclusive) to the 20th of March, belonged to the first month, and were, accordingly, true Paschal Moons, in so far as they fulfilled the condition respecting the Equinox. But, on the other hand, as these Moons occurred before the 21st of March, to which day the ancient Canons

assuming the exactness of the Julian year, fixed the Equinox (Art. 41), they were *not* true Paschal Moons, and were, consequently, always rejected by the Church as belonging not to the *first* month of the year, but to the *last* month of the preceding year. But in thus avoiding one violation of the ancient Rules, another was unavoidably incurred—namely, that of celebrating Easter in the *second* (instead of the *first*) month, whenever a fourteenth Moon followed March 10 and preceded March 21: because these fourteenth Moons, not being Paschal Moons *in relation to March* 21, did not become so until the following month, which was the *second*. The Golden Numbers giving rise to this contradiction were, as the Paschal Table (Art. 60) shows, III., VI., VIII., XI., XIV., XIX. Thus, in process of time, all the regular Paschal Moons (as regards the true *Equinox*) would have been rejected, and Easter would be celebrated in the second, third, &c., month, and even in the height of Summer, or the end of Autumn. When the Equinox should have receded 29 days, or more, from the 21st of March—namely, after the year 4100 (1)—Easter would cease to be even celebrated in the *first month*, contrary to the law of the Ancient Church.

Again, on account of the four days' error in the Lunar Cycle, it often happened that Easter was celebrated on the twenty-fifth day of the Moon—namely, whenever the fourteenth day of the Moon, found by the Golden Number, and which was, in truth, the eighteenth day from mean New Moon, fell upon Sunday, and therefore Easter was deferred to the following Sunday. This is entirely opposed to the ancient usage of the Church, which again and again inculcated that Easter should be celebrated from the fourteenth Moon of the first month to the twenty-first, both inclusive.

(1). The reason of this cannot be fully understood, until we come to the explanation of the "Solar Equation." For the present, it is sufficient to say that the Rule for finding how many days (in addition to the 10 days omitted at the Reformation in 1582) must be omitted, on account of the *Solar Equation*, up to any given centurial year A. D., is the following:—

"Subtract 16 from the number of centuries in the given year: divide the difference by 4: multiply the quotient by 3, and add the remainder to the product. This will give the required number." *E.g.*, let the given year be A. D. 4100: we have

$$\frac{41 - 16}{4} = 6, \text{ leaving 1 over,}$$

and

$$6 \times 3 + 1 = 19 \;\therefore\; 10 + 19 = 29.$$

The reason of this Rule is (as we shall see hereafter) that in every four centuries, reckoning from A. D. 1600, three intercalary days are omitted: hence the quotient multiplied by 3 gives the number of omitted days due to the groups of 400 years: while the remainder, which is always less than 4, consists of *Common* centurial years, which, therefore, must be added to the aforesaid product (*Clarius*, pp. 134 and 655).

The above is on the supposition that the Gregorian Rule, of dropping 3 days in 400 years, be adopted. The exact date when, supposing the error of the Julian year to be 1 day in 128 years, it would amount to 29 days, is found by the proportion $1^d : 29^d :: 128^y : 3712^y$, to which, if we add 325 (date of Nicene Council), we get 4037 A. D.

75. From what has just been said, we can readily believe the fact that, from the year 1500 to 1582—*i. e.*, in the comparatively short space of 83 years—no less than 54 Easters were wrongly celebrated; and that, had not the Calendar been reformed, there would have been 12 more wrong celebrations during the remaining 17 years of that sixteenth century. Of these wrong celebrations in the sixteenth century, some were 7, some 28, some even 35, days later than the Ancient Rule required. The error of 35 days occurred in 1565, 1568, 1576, 1579; and it would have occurred again in 1595 and 1598. Indeed, Clavius has shown, by actual calculation, that, had not the Old Calendar been reformed, there would be only 265 *legitimate* Easters during the 3401 years from A. D. 1600 to A. D. 5000; the *remaining* 3136 *being contrary to the ancient Canons* respecting Easter. Of this total number, 581 would be wrong by 35 days; 881 by 42 days; and 49 by 49 days; the remainder being wrong by 7, 14, or 28 days. *In fact, after the year* 2698, *there would be no legitimate Easter.* We may put the same thing in another form, by saying that the coincidence between the Old- and New-Style Easter Day, which still happens occasionally, and will continue to happen, though less and less frequently, will occur for the last time in the year A. D. 2698 ([1]).

(1). The reason of this will be shown hereafter (Art. 151). On the subject of this and the foregoing paragraph, vid. *Clavius*, cap. ii. §§ 1–3; cap. xxii. pp. 414, *sq.*; and, especially, p. 464.

76. We come now to describe the mode whereby the accumulation of errors, resulting from the inexactness of the Julian year, and from that of the Lunar Cycle, was got rid of in the Gregorian Reformation. First, as regards the 10 days arising from the over-length of the Julian year. These 10 days might have been wiped out in one of two ways: either (after the example of Augustus, in the case of the mistake respecting the Julian rule of intercalation, Art. 18) by dropping a Bissextile day every fourth year, for 40 years; or by suppressing at once the whole of the 10 days. Lilio suggested both these ways, but recommended the latter. I have already mentioned (Art. 73) that the Pope sent a *Compendium* of Lilio's plan to the most celebrated academies of Europe, for their criticism and judgment upon it. Some advised that the 10 days should not be dropped at all; but that the Equinox should, for the future, be fixed to the 10th or 11th of March, on which days it then actually fell ([1]); just as by the old Paschal Canons it was fixed to the 21st of March, where it actually was about the time of the Nicene Council. This

was a very obvious and simple expedient. But a too scrupulous regard for the traditional day prevented its being adopted, as it ought to have been. Had this been done, all the confusion arising from *difference of style*, and most of the chronological errors which have embarrassed some countries ever since, would have been avoided. Others, again, recommended that if the accumulated days were to be dropped at all, not merely the 10 which had accrued since the year A. D. 325 should be omitted, but 2 or 3 more; 2, so as to restore the Equinox to the place (23rd of March) which it occupied, in the Julian Calendar, at the Birth of Christ; 3 (as Lilio suggested), to move it to its place at the time of the Julian Reformation, B. C. 45 (2). Gregory's mathematicians elected to drop 10 days at once, so as to restore the Equinox to the day (March 21) on which it fell about the time of the Nicene Council. Accordingly, it was decreed in the Pope's Bull (3) that the 10 days following the 4th of October (the Feast of St. Francis), 1582, should be wholly omitted from the Calendar; and that the day next after the 4th should be deemed and called the 15th of October: so that the days of that month should be reckoned thus, 1, 2, 3, 4, 15, 16, &c.; the month thus being made to consist of only 21 days.

The reason why the month of October was selected for the suppression of the 10 days was because this month was most free from Saints' days, and therefore the omission would cause less interference with usage. The Sunday Letter of this year being G, the omission of the 10 days changed the Letter to C for all the Sundays following the 15th, to the end of the year (4). Thus the year 1582, though not a Leap-year, had two Sunday Letters, G and C: and it contained only 355 days.

The omission of the 10 nominal days in 1582 also necessarily caused that all the Calendar New Moons, from the 5th of October to the end of that year, fell 10 nominal days later than before the correction; in other words, the Golden Numbers coming after the 4th of October that year, and throughout all the months in succeeding years, ceased to indicate the Calendar New Moons as before; and, in order that they should do so, those after the 4th of October, in 1582, had to be moved *down ten* places; and in the following years all of them should be so moved, were the Solar error alone to be corrected. But we shall see presently that, in consequence of the Lunar correction taking place in the *contrary direction*, the Golden Numbers were actually moved down only 7 places.

(1). The Equinox, in consequence of the Leap-year, vibrates between two days.

(2). On this point Sir John Herschel observes (Astr., edit. xi. p. 690):—"The Reformation of Gregory was, after all, incomplete. Instead of 10 days he ought to have omitted 12. The interval from January 1, A. D. 1, to January 1, A. D. 1582, reckoned as Julian years, is 577460 days, and as Tropical, 557448, with an error not exceeding 0·01 days. The difference being 12 days, their omission would have completely restored the Julian Epoch. But Gregory assumed for his fixed point of departure, not that epoch, but one later by 324 years—viz., January 1, A. D. 325; assuming which, the difference of the two reckonings is $9^d{\cdot}505$; or, to the nearest whole number, 10 days." *Vid.* Note 4, Art. 72, where the number is calculated to be 9·7945 days. According to Delambre, the Vernal Equinox fell, B. C. 45, on March 23; A. D. 325, on March 20; and A. D. 1582, on March 11 (*Ideler*, i. p. 78).

(3). Vid. *Clavius*, p. 14.

(4). Ten Calendar Letters were passed over. But 10 = 14 – 4; therefore, the Calendar Letter of the 15th was 4 places higher up than that of the 4th. So that, as G was the Sunday Letter that year, the next Sunday Letter after the 4th was 4 places higher up than G: *i. e.*, C. Or thus: the Calendar Letter of October 4 is D; therefore (G being the Sunday Letter that year) October 4 was Thursday: the next day, Friday, was reckoned as the 15th, the Calendar Letter of which is A; consequently, Saturday, 16th, B; Sunday, 17th, C.

77. Thus, the accumulation of error arising from the overlength of the Julian year was wiped out. The year 1582, in which the correction was made, was the second "*Annus confusionis.*" The next point to be settled was, how best to prevent any such accumulation for the future; or, to state the same thing in other words, how the Calendar Equinox, which was still to be fixed to the 21st of March (1), might be kept as close as possible to the true Equinox. According to the Alphonsine Tables, which were followed by Gregory's mathematicians, the length of the Tropical year was $365^d\ 5^h\ 49^m\ 6^s$; that is to say, $10^m\ 44^s$ shorter than the Julian year. The difference amounts to a day in about 134 years, or to 3 days in 402 years. Accordingly, the plan adopted was to omit 3 Bissextile days every 400 years. This was done by making the years 1700, 1800, 1900, *Common* years, leaving 2000 a Bissextile; and so on, for every successive group of four centuries. The Gregorian Rule of intercalation may, therefore, be thus generally expressed:—

"*The Rule is the same as in the Julian Calendar, except that the centurial years (centesimi anni) not divisible exactly by* 400 *are to be Common years.*"

We have thus an exact analogy between the ordinary Julian intercalation of one day every 4 years and the Gregorian correction, which intercalates a day in those centesimal years only which are measured by 4 (2).

This mode of preventing, for the future, the accumulation of error arising from the inexact length of the Julian year is called the *Solar Secular Equation.*

(1). The centurial year 1600 was allowed to remain unchanged, because, being so close to 1582, when the accumulation was cleared off, it did not need correction.

The principle of this elegant artifice was not due to Lilio or Clavius. Stöffler, in his "Calendarium Romanum," 1518, had proposed the omission of one day in 134 years. But Pitatus of Verona, in a petition presented to the Council of Trent in 1552, proposed that 1600, 1700, 1800, . . . 2000, 2100, 2200, . . . should be Common years. The groups of three consecutive *Common* centesimal years were his suggestion. The Gregorian correctors merely changed slightly the centesimal years so grouped.

(2). *Clavius* (p. 86) states that the reason why *centesimal* years were chosen for making the Solar Equation, rather than dropping the intercalation as often after the year 1600 as the error amounted to one day (according to him in 134 years), was this: that, as 134 is not divisible by 4, the omission should be made sometimes after 132 years, sometimes after 136: "sed quoniam varietas hæc, quâ, modo anno quolibet 132, modo quovis anno 136, intercalatio omitti deberet, facile oblivioni tradi possit, quod anni hujusmodi nullâ insigni notâ sint affecti, electi sunt ad hanc rem anni potius centesimi, cum hi magis sint conspicui, et non tam facile in illis negligentia aliqua aut oblivio possit obrepere; præsertim quod omnes centesimi sunt Jubilæis dedicati, et propterea notissimi et expectatissimi."

As to retaining the intercalation every 400th year, instead of every 402nd, as it strictly ought to be ($134 \times 3 = 402$), he further observes that the error is so very slight (amounting to only one day in 26800 years), that it is not worth while to disturb the regular intercalation for it. After the lapse of 26800 years from 1600 (A. D. 28400), a day may be intercalated in some Common year to correct the error; that is to say, to bring back the Equinox from the 22nd to the 21st of March (*Clavius*, p. 205). The error in question is thus calculated:

$$3\left(\frac{1}{400} - \frac{1}{402}\right) = \frac{6}{160800} = \frac{1}{26800}.$$

Or thus (Clavius): if, in 134 years, 1 intercalary day be omitted, in 2 years $\frac{1}{67}$th of a day is omitted: therefore, $\frac{1}{67}^{d} : 400^{y} :: 1^{d} : 67 \times 400 = 26800$ years.

It has been objected that the omission of the intercalary day for 3 consecutive centurial years, instead of omitting it at the end of each 134th year, is not so well suited as the latter would be to retain the Calendar Equinox in its place, inasmuch as the exact Equation is anticipated—the first century, by 34 days; the second, by 68 days; and the third, by 102 days. But it is easy to show that the contrary is the fact: that is to say, if the omission of the day were made every 134th year, the Equinox would *always* recede from its assigned place (March 21) towards the beginning of the month, and that the regression would amount to nearly a day at the end of the 134th year; and this would occur *three* times in the space of 400 years: whereas by the centesimal omission, the Equinox will not recede more than 18 hours from its proper place; and *that* only *once* in the space of 400 years.—Vid. *Clavius*, cap. vi. § 18.

78. The Gregorian omission of 3 days in 400 years supposes the year to consist of $365^{d} + \frac{97}{400}$; that is, of $365^{d}\ 5^{h}\ 49^{m}\ 12^{s}$, which is nearly 24^{s} too long. This would amount to a day in 3600 years (1). Accordingly, Delambre proposed (2) that the year A. D. 3600, which, according to the Gregorian Rule, is to be a Leap-year, and also its multiples, 7200, 10800, &c., should be Common years. By this modification the Calendar would be brought into very close accordance with the Sun. If the Gregorian Rule remain unmodified, the Calendar Equinox will, in the course of about 35,000 years, deviate as much from the true Equinox as it did in the year 1582: while, in the *uncorrected* Julian Calendar, the Easter Festival will then have nearly run through the whole circle of the seasons.

Sir John Herschel has proposed (Astron. p. 673) a correction of the Gregorian Rule, slightly different from that of Delambre—viz., by extending the Gregorian Rule one step farther, and making the years divisible by 4000 Common years. This is equivalent to reckoning 969 Leap-years in 4000 years, so that the year would consist of $365^d + \frac{969^d}{4000} = 365\cdot24225$. The fraction $= 5^h\ 48^m\ 50\cdot4^s$. If with him we suppose the Tropical year to be $365^d\cdot24224$, this correction would make the Gregorian Calendar to differ from the true Solar year by only one day in 100,000 years [3].

If the Solar Equation only were in question, a simpler and more exact correction than the Gregorian would be to suppress one Leap-year every 128 years; in other words, to make every 128th year drop its intercalary day. This would make the length of the year $365^d + \frac{31^d}{128} = 365\cdot2421875$; which differs from $\cdot242216$ by only $\cdot000028$ of a day, which would amount to a single day in 35,714 years [4]; which is nearly ten times as accurate as the Gregorian correction of omitting 3 days in 400 years. But, in the first place, the omission of 3 consecutive centurial days is more remarkable and better defined than the omission of one day every 128th year (*vid.* Note 2, Art. 77); and, secondly, there would be no neutralizing of the Lunar Equation, such as takes place in the Gregorian plan (*vid.* Arts. 81, 82).

(1). The authors of the Gregorian Calendar adopted as the length of the Tropical year that given in the Alphonsine Tables, viz., $365^d\ 5^h\ 49^m\ 16^s$, *i. e.*, $10^m\ 44^s$ shorter than the Julian year, which would amount to a day in about 134 years, or 3 days in 400 years. Reducing to decimals we have

Julian year, $365^d\ 6^h$	$= 365\cdot25$
Alphonsine year, $365^d\ 5^h\ 49^m\ 16^s$	$= 365\cdot2425462$
Difference	$= \quad \cdot0074538$,

which fraction, multiplied by 400, $= 2\cdot98152$, which is less than 3 days by only $\cdot01848$ of a day. The Gregorian correctors neglected this fraction, and assumed the difference to be 3 days exactly, and accordingly suppressed 3 Bissextiles in 400 years. In other words, the length of the Gregorian year

$$= 365^d + \frac{97^d}{400} = 365^d\cdot2425:$$

which exceeds the true Tropical year ($365\cdot242216$) by $\cdot000284$ of a day; and this fraction would amount to a day in about 3520 years.

(2). Delambre, Astron. Mod., i., pp. 8, 696. This proposal was adopted in the French Republican Calendar (1792). Strictly speaking, this correction should not be made until 3600 after A. D. 1600, that is to say, in A. D. 5200; and then, after that, every 3600th year.

(3). $$\frac{969}{4000} = \frac{1000 - 30 - 1}{4000} = \frac{1}{4} - \frac{3}{400} - \frac{1}{4000},$$

which is the extended Gregorian Rule.

But if the exact length of the Tropical year be $365^d \cdot 242216$ (Art. 11), the difference between it and the length assumed by Sir John Herschel's proposed modification is ·000034; that is to say, the Gregorian Calendar, corrected as he proposes, would err one day (in excess) in about 29,000 years.

(4). Sir E. Beckett (Astron. without Math., 5th ed., p. 152) strongly recommends this plan. Francœur also suggested it, but did not recommend it (Theorie du Calendre, p. 299). The former puts the question thus:—A million Julian years = 365,256,000 days; but a million Tropical years = 365,242,216 days. Therefore, the problem is how to drop 7784 Leap days, in some neat and simple way; that is, one day in 128·47 years.

79. It may be interesting to ascertain whether there are any other possible methods of restoring the coincidence of the Civil year of 365 days with the Tropical, and which may be more exact, and at the same time equally or more convenient than the (Julian and) Gregorian intercalations. I have accordingly, in a note, given the mathematical solution of this question (1). The result is that there are several such methods—some of them more exact than the Gregorian, but most of them inconvenient in practice. So that, in fact, the Gregorian method, extended in the way proposed by Delambre, or by Sir J. Herschel, seems to answer sufficiently all practical purposes.

Before passing on from this subject, I may just notice another mode of intercalating, by *centuries*, which in some respects seems preferable to the Gregorian. I mean dropping the intercalary day of every century except the *fifth*, beginning with the sixteenth; instead of the *fourth*, beginning with the seventeenth; in other words, making 1600, 1700, 1800, 1900 Common years, and 2000 a Leap-year, and so on. This would make the year

$$365^d + \frac{121^d}{500} = 365 \cdot 242^d,$$

which differs from ·24224 (Herschel's Tropical year) by ·00024, which would not amount to quite a day in 4000 years. With the lesser and more correct fraction ·242216, the difference would be only ·000216, which would amount to a day in about 5000 years.

(1). To find mathematically the different possible modes of intercalation by which the Civil year of 365 days may be kept approximately close to the Tropical year, we have only to convert the fraction expressing the excess of the Tropical year over 365 days into a continued fraction, and then find the series of resulting convergent fractions.

Let us take, first, the fraction of a day adopted in the Nautical Almanac, viz., $\cdot 242216 = 5^h\ 48^m\ 47 \cdot 46^s$.

This, expanded into a continued fraction, gives

$$\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{7+\cfrac{1}{1\ \&c.}}}}}}}}$$

The series of convergent fractions is

$$\frac{1}{4},\ \frac{7}{29},\ \frac{8}{33},\ \frac{31}{128},\ \frac{39}{161},\ \frac{70}{289},\ \frac{529}{2184},\ \&c.$$

These fractions are, by the nature of the continued fraction, alternately greater and less than the true value of the original fraction. But the intercalation represented by each is the most exact possible within the number of years indicated by the denominator.

The first approximation, $\frac{1}{4}$, is too great: it is the Julian intercalation of one day every 4 years.

The second approximation, $\frac{7}{29}$, errs by defect. It gives 7 intercalations in 29 years; that is, 6 ordinary Leap-years, in succession, and a seventh intercalary year at the end of the next *five* years. It supposes the Tropical year to be 365 $\frac{7^d}{29}$, = $365^d\ 5^h\ 47^m\ 35{\cdot}17^s$, which is too short by $1^m\ 12{\cdot}29^s$.

The third approximation, $\frac{8}{33}$, is greater than the true value. It gives 8 intercalary years in 33 Common years; viz., 7 ordinary Leap-years, in succession, and an eighth intercalary year at the end of the next *five* years. It supposes the Tropical year to be 365 $\frac{8^d}{33}$, = $365^d\ 5^h\ 49^m\ 5{\cdot}45^s$, which *exceeds* that of the Nautical Almanac by only 18^s; whereas the Gregorian year ($365^d\ 5^h\ 49^m\ 12^s$) exceeds it by $24\frac{1}{2}^s$. Accordingly, this mode of intercalation produces a nearer coincidence between the Civil and Tropical year in 33 years, than the Gregorian method does in 400 years: and, moreover, on account of the shortness of its period, it confines the deviation of the nominal Equinox from the true within much narrower limits. This approximation $\left(\frac{8}{33}\right)$ is said to have been proposed by Omar, a Persian astronomer, A. D. 1079, five centuries before the Gregorian reformation. The accumulation of error arising from it would amount to only one day in 4800 years. The chief defect of it would be that it would derange the intercalary years, making some of them fall in *even* years, A. D. and some in *odd*. Then, to take the first century, we should have the intercalary years

0, 4, 12, 16, 20, 24, 28, 33; 37, 41, 45, 49, 53, 57, 61, 66; 70, 74, 78, 82, 86, 90, 94, 99.

It is true that it would only be necessary to bear in mind that the retarded Bissextiles form an arithmetical progression, 0, 33, 66, 99, 132, &c., the ordinary ones occurring at the usual interval of 4 years. Still, the Gregorian system is, on the whole, the most convenient.—*Delambre*, Astr. Mod. i., p. 78.

The fourth approximation, $\frac{31}{128}$, is less than the exact value, but it is extremely accurate. It consists in dropping the intercalary day in every 128th year; or in other words, making only 31 Leap-years instead of 32. It supposes the Tropical year to be 365 $\frac{31}{128}$ days = $365^d\ 5^h\ 48^m\ 45^s$, which is *less* than the true fraction by only $2\frac{1}{3}^s$; which would amount to only a single day in about 34,500 years. It would, however, be very inconvenient, on account of the complexity of the rule of intercalation. For $\frac{31}{128} = \frac{3 \times 8 + 7}{3 \times 33 + 29}$; so that there should be really four periods; three of 33 years each, containing 8 Leap-years a-piece; and one of 29 years, containing 7 Leap-years. The eighth approximation, $\frac{520}{2184}$, differs from the original fraction by only about the 100th part of a second, but is wholly unsuited for actual use.

Let us now take the length of the Tropical year as found by Lalande, and adopted by Ideler and other eminent authorities, and which is only about half a second longer than that which we have been just considering. I mean $365^d\ 5^h\ 48^m\ 48^s = 365^d{\cdot}2422222$. Converting the decimal part into a continued fraction, we get

$$\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{2}{1+0.}}}}}}$$

The series of converging fractions is

$$\frac{1}{4},\ \frac{7}{29},\ \frac{8}{33},\ \frac{31}{128},\ \frac{39}{161},\ \frac{70}{289},\ \frac{109}{450}.$$

The first six agree precisely with the first six found above. The seventh, $\frac{109}{450}$, is exact. It $= \frac{872}{3600}$, giving 872 intercalations in 3600 years, in place of 900 Julian intercalations, and 873 Gregorian. The latter therefore becomes exact by omitting the Bissextile every 3600th year, reckoning from the year 1600. In fact,

$$\frac{109}{450} = \frac{872}{3600} = \frac{900-28}{3600} = \frac{900-27-1}{3600} = \frac{1}{4} - \frac{3}{400} - \frac{1}{3600}.$$

The two first terms are the Gregorian intercalation; and the third is the correction necessary to make the Civil Calendar agree completely with the year $365^d\ 5^h\ 48^m\ 48^s$.

Upon the whole, this seems to be the simplest mode of correcting the Gregorian intercalation; and it has the additional great merit of not interfering with the *Lunar* Equation.

According to *Delambre*, the length of the year = $365^d\ 6^h\ 48^m\ 51{\cdot}6^s = 365^d{\cdot}2422642$, which decimal developed into a continued fraction gives the series

$$\frac{1}{4},\ \frac{7}{29},\ \frac{8}{33},\ \frac{39}{161},\ \frac{47}{194},\ \frac{321}{1325}.$$

The Gregorian Correction, $\frac{97}{400}$, is a combination of the first and fifth; for, doubling the fifth and trebling the first, we get $\frac{94}{388}$ and $\frac{3}{12}$; and $\frac{94+3}{388+12} = \frac{97}{400}$.

For fuller details on this subject of Intercalation, *vid.* Delambre, Astron. Mod., tom. i., pp. 73, *sq.*

80. Let us now consider the Gregorian reformation in reference to the error of the *Lunar Cycle.* I have already said (Art. 72) that the real (or, rather, the mean) Moon was then more than 4 days in advance of the Calendar Moon; that is to say, the Calendar Moon by which Easter was regulated fell more than 4 days later in the month than the mean Moon of the heavens. This *anticipation* of the Calendar Moon by the actual mean Moon was called by the Computists the προέμπτωσις of the Moon (1). Gregory's mathematicians calculated the amount of it to be about one day in 312½ years (2); which in 1257 years (1582–325) amounted to over 4 days, and in process of time, if uncorrected, would, of course, gradually give rise to still greater and greater divergences between the mean and Calendar Moons. In fact, about the year A. D. 5000, the divergence would be such that the Calendar New Moon would occur at the time of the actual Full Moon, and *v. v.;* so that the absurdity would arise that Easter would be celebrated about the time of the actual *New* Moon.

As the Golden Numbers in the Old Calendar denote the *Calendar* New Moons, and as the *mean* New Moons, at the time of the reformation by Gregory, fell on the fifth day higher up in the month than the Calendar Moons (Art. 72), the obvious mode of getting rid of the accumulated error was to move all the Golden Numbers 5 days higher up (3 vacant days being left between the old and new places), so as to bring the Calendar New Moons into a near agreement with the mean Moons. We have already seen that, in order to restore the true Equinox to the 21st of March, 10 nominal days were dropped in 1582 (the 15th of October following immediately after the 4th); the effect of which would be that all the remaining Calendar New Moons that year, indicated by the Golden Number VI. (which was the Golden Number of the year), would fall 10 whole nominal days later than in the Old Calendar; that is to say, as the Golden Number VI. was affixed to October 20, November 19, and December 18, the Calendar New Moons, after the omission of the 10 days, would fall on October 30, November 29, and December 28; and, in order to indicate those New Moons, Golden Number VI. should have been moved down those 10 places, and, with VI., all the rest of the Golden Numbers should also be moved, their *relative* position remaining unchanged. But, as we have seen (Art. 72), all the Calendar New Moons (or Golden Numbers) were already (in consequence of the accumulated error of the Lunar Cycle) 4 days lower down than the mean

New Moons (3 vacant days being between them); and to wipe out this error, it was necessary to put them 4 days back, or higher up; thus leaving the actual downward displacement of the Golden Numbers, from both causes combined, only *six*. But Clavius preferred, for the reason stated in the Note (3), to make them *ascend* (on account of the Lunar error) only *three* days, instead of four; so that, as the final result of both corrections, the Golden Numbers were moved down *seven* days from their old places; that is to say, VI. (the Golden Number of 1582) was moved down from its old place (October 20) to October 27 (6 vacant places intervening). Accordingly, in the amended Calendar, after 1582, had the Golden Numbers been retained, as before, to indicate the Calendar New Moons, instead of being replaced (as they were) by the Epacts, all the 19 Golden Numbers should have been moved down 7 days from their old places. So that, for example, III., which in the Old Calendar was affixed to January 1, would in the New Calendar be affixed to January 8, six vacant days intervening between the new and old places: and so of all the rest. We shall return to this point, when we come to consider the Paschal Table as it stood after the Gregorian reformation (Art. 105).

(1). Similarly, the falling behind (or lower down in the month) of the nominal Equinox (March 21) from the true place of the Equinox (Art. 72) was termed 'metemptosis' (μετέμπτωσις). These terms were applied to the Solar and Lunar Equations, respectively. (*Delambre*, Astr. Mod. i., p. 15).

(2). Gregory's mathematicians, following the Alphonsine Tables, assumed the mean length of a Lunation to be a fraction of a second more than $29^d\ 12^h\ 44^m\ 3^s$, which multiplied by 235 gives $6939^d\ 16^h\ 32^m\ 27^s$, differing from 19 Julian years by $1^h\ 27^m\ 33^s = 5253^s$. Hence, $5253^s : 19^y :: 86400^s : 312\frac{1}{2}^y$, *q.p.*

The Gregorian correction for the proemptosis is, accordingly, 8 days in 2500 years. But the more exact correction is one day in 308 years (Art. 72), or 8 days in 2464 years. Hence, according to the Gregorian correction, the Calendar New Moons would fall one day too late in about 21,300 years. The proportion is, 36 : 2500 : : 308 : 21355.

Clavius himself (p. 156) calculates the error of his correction to be only 8 hours in 481,436 years, and he naïvely adds, "relinquamus eam curam posteris nostris, si tamdiu mundus durabit."

(3). The Jews celebrated their Passover, and the Quartodecimans their Easter, on the 14th day of the real, astronomical, Moon; that is to say, the day before the real Full Moon, which falls about the fifteenth day after conjunction. Now, the great object of the Ancient Easter Canons (Art. 41) was that this Festival should be celebrated as nearly as possible at the same time of the year as the original event which it commemorated took place; that is, at the same time of year as the Jewish Passover; not, however, *before* the Passover, because it would be absurd to commemorate the Resurrection before the Passion; nor on the *same day* as the Passover, for this would be to symbolize with the hated Jews, and to adopt the Easter practice of the heretical Quartodecimans: but as nearly as possible *after* the Passion. In order to avoid celebrating Easter *before* the Passover, it was necessary so to arrange the New Moons of the cycle, or Calendar New Moons (indicated by the Golden Numbers) that the 14th day of the Paschal (or Easter) New Moon should not fall *two* or more days before the 14th day of the real (or mean) astronomical Moon, which determined the Jewish Passover; otherwise, Easter Day might under certain circumstances be celebrated *before* the Passover day. For example, if the 14th day of the Calendar Pascha Moon were the 12th day of the real Moon, and if the following day were Sunday, then Easter Day would be kept on the 13th of the real Moon, and, therefore, before the Passover. Again, in order that Easter Day should not fall *on* the Passover day, it was necessary that the first day of the Calendar Paschal New Moon should not fall on the *same* day as the real (or mean) astronomical New Moon: for if it did, it might sometimes happen that Easter Day would be kept on the Passover day. For example, if the mean New Moon were to fall on March 8, at noon, the mean Full Moon would fall about 6 A. M. of the 23rd (the interval between mean conjunction and opposition being nearly 14^d 18^h 22^m), and the actual Full Moon would also often fall on that day; but the 14th day of the Calendar Paschal Moon would be the 21st; and if the 22nd were Sunday, Easter Day would be kept on the 22nd, a day before the actual Full Moon, and, therefore, on the Passover day. Lastly, if the Calendar New Moons, or Golden Numbers, be so adjusted that they shall fall one day, at least, *later* than the actual, or mean New Moon, then the 14th day of the Calendar Paschal Moon will fall on the 15th of the real, or mean, astronomical Moon, that is to say, on the day of the real Full Moon, and, therefore, *after* the Jewish Passover day; and if this day of Full Moon be also a Sunday, Easter Day might, in accordance with the fundamental principles of the Old Calendar, be kept that day: but in point of fact, it was not kept until the Sunday after, in compliance with the general Rule laid down in Canon II., Art. 41; which Rule was obviously framed to meet the case, before mentioned, of the 14th day of the Calendar Moon falling on the 14th day of the real Moon (the Passover day), when the week-day should happen to be Sunday; for by this Rule Easter Day would be postponed to the following Sunday. Hence we see the reason of what Clavius says in the passage above referred to (cap. xviii. 4), "The New Moons of the cycle (indicated by the Golden Numbers in the Julian Calendar, and by the *Epacts* in the Gregorian Calendar), should be so arranged that the 14th day of a Paschal Moon should never *precede* mean opposition (the mean Full Moon) by two or more days; but either by one day only, or else fall on the day of mean Full Moon, or not long after:" and he adds, "we have endeavoured with the utmost care and diligence to adjust our cycle (of Epacts) so that the Paschal New Moons of the Calendar shall follow the real (or mean) New Moons, so that the 14th of the Calendar Moons should fall either the day *before* the real (or mean) Full Moon, or *on* that day, or not long after." And hence we also see *the reason why, in clearing of the accumulated error of the Lunar Cycle, he did not put back the Golden Numbers four days higher* than their old places (which would have made the Calendar New Moons coincide with the real New Moons) *but only three*, whereby the Calendar New Moons were made to fall one day later than the real, or mean, Moons.

81. Having seen how the accumulation of error arising from the inexactness of the Lunar Cycle was cleared off, we have next to inquire (as we have done in the case of the Solar Secular Equation) by what mode a similar accumulation in future was prevented; in other words, to explain the *Lunar Secular Equation*. Gregory's mathematicians calculated (Art. 80) that the Moon of the cycle, or Calendar Moon, dropped behind the real astronomical Moon one day in 312½ years, or 8 days in 2500 years. The plan, therefore, which they adopted, in order to rectify this divergence between the Calendar New Moon and the mean astronomical New Moon as fast as it should arise was, *to advance one day higher up* (towards the beginning of the month) all the Calendar New

Moons *every third centurial year*, reckoning from A. D. 1800, seven times consecutively; and to defer the eighth advance of a day until the fourth centurial year after the seventh advance; thus making an aggregate correction of 8 days every 2500 years, beginning from 1800. In this way, supposing the proemptosis of one day in 312½ years to be exact, the error of the Lunar Cycle would be very nearly corrected every 300th year after 1800; and completely so at the end of the cycle of 2500 years. It would then recommence, and be similarly wiped out at the end of the next cycle; and so on perpetually.

The year 1800 itself was taken as the first Lunar Equation year after the correction (1). Consequently, the first period of 2500 years contained *nine* Lunar Equation years, while all the subsequent periods contained but *eight*. Hence the scheme of the Lunar Equation was the following, the sign + denoting the centurial years in which the *Golden Numbers* were all advanced one day higher, and the *Epacts* were all increased by unity:

1800 + 1		
═		
2100 + 1	4600	7100
2400 + 1	4900	7400
2700 + 1	5200	7700
3000 + 1	5500	8000
3300 + 1	5800	8300
3600 + 1	6100	8600
3900 + 1	6400	8900
4300 + 1	6800	9300
═	═	═

and so on.

(1). The *accumulated* error having been wiped out in 1582 by the upward movement of the Golden Numbers (Art. 80), the amount of error in the following 218 years would be only about two-thirds of a day, and therefore it would seem that the first Equation should more properly have been made in 1900. But Clavius preferred 1800 for a reason which we shall see hereafter, when speaking of the *Epacts* (*vid.* Art. 128).

82. Having now explained the nature and effects of the Solar and Lunar Equations *separately*, let us consider their combined result as regards the displacement of the Golden Numbers in the Calendar.

The effect of the *Solar Equation*, or dropping of the Bissextile day for the centurial years before mentioned (Art. 77), being to *lower* all the Golden Numbers one day; and the effect of the Lunar Equation being, as we have just seen, the contrary, that is, to *raise* them all one day; it follows, of course, that, when the *Solar Equation alone* takes place, the Golden Numbers *descend* one place; when the *Lunar Equation alone* takes place, they *ascend* one place; when *neither* takes place, or *both* concur, there is no displacement in the Golden Numbers. As the year 1600 was allowed to remain a Leap-year (being too close after the correction in 1582 to allow of any fresh accruing of the Solar error), there was no Solar Equation; and the Lunar Equation had not yet commenced. There was, consequently, no change made in the position of the Golden Numbers during the 17th century (1600–1699), but they remained in the same position as they occupied in 1583. In 1700 the Solar Equation took place (for the first time), requiring the lowering of the Golden Numbers one day during the 18th century (1700–1799). There was no Lunar Equation that year. In 1800, the Solar Equation again took place; and also the Lunar Equation (for the first time): consequently, they neutralized each other, and no change took place in the position of the Golden Numbers; but they have continued, and will continue during the 19th century (up to 1899), to occupy the same places they did during the 18th. In 1900 the Solar Equation will occur, and there will be no Lunar Equation; hence, during the 20th century (1900–1999), the Golden Numbers will require to be moved down one place—making the total displacement, from the date of the reformation of the Calendar, *two* places downwards. In the year 2000 there will be neither Solar nor Lunar Equation, and accordingly no change will be made during the 21st century (2000–2099); but the Golden Numbers will occupy the same places as they did during the 20th century. In the year 2100, both Equations will take place; and, therefore, as they neutralize each other, no change will be made in the Golden Numbers, but they will continue during the 21st century (2100–2199) as they were during the 19th and 20th centuries. Instead of pursuing these details any further, I have subjoined the following *Table*, showing the separate and combined effects of the Solar and Lunar Equations for the different centuries after 1600, with the resulting number of *displacements of the Golden Numbers downwards*, corresponding to each century. That the *general tendency* of the displacements will be *downwards* is obvious; because in 24 centuries there are 18 Solar Equations (or, 18 *downward* displacements); while in 25 centuries there are only 8 Lunar Equations, or 8 *upward* movements. As the Solar and Lunar usually take place in the same century, and so balance each other, the number of actual regressions due to the Lunar Equation is very small; *e. g.*, A. D. 2400, 3600, 5200, 6400, 8000, 9600, 10800; the law of which progression is evident. (*Vid.* Art. 84.) It must be borne in mind, also, that the Equation in both cases, Solar and Lunar, is made in the centurial year, and applies to all the rest of the century; *e. g.*, from 1600–1699, both inclusive, there is no change; from 1700–1799, both inclusive, there is a displacement of *one* day, downwards. From 1800–1899, there is no change, and the downward displacement continues to be *one*. But from 1900–1999, there is another Solar Equation, and a fresh displacement takes place, making the total number, for that century, *two;* and so on. The effect of the Solar Equation being to move the Golden Numbers down, or to *increase* their distance from the beginning of the month, this Equation has the positive sign before it: the effect of the Lunar Equation being, on the other hand, to raise the Golden Numbers up, or diminish their distance from the beginning of the month, the negative sign is prefixed.

83. EQUATION TABLE

SHOWING THE SOLAR AND LUNAR EQUATIONS, SEPARATELY, WITH THEIR COMBINED EFFECT, AND THE TOTAL DISPLACEMENT THENCE RESULTING.

YEARS A.D.	Solar Equation.	Lunar Equation.	Total displacement of Golden Numbers downwards.	YEARS A.D.	Solar Equation.	Lunar Equation.	Total displacement of Golden Numbers downwards.	YEARS A.D.	Solar Equation.	Lunar Equation.	Total displacement of Golden Numbers downwards.	YEARS A.D.	Solar Equation.	Lunar Equation.	Total displacement of Golden Numbers downwards.
D 1600	0	0	0	k 4600	+ 1	- 1	13	H 7600	0	0	26	p 10600	1	0	39
C 1700	+ 1	0	1	i 4700	+ 1	0	14	H 7700	1	- 1	26	n 10700	1	0	40
C 1800	+ 1	- 1	1	i 4800	0	0	14	G 7800	1	0	27	p 10800	0	- 1	39
B 1900	+ 1	0	2	i 4900	+ 1	- 1	14	F 7900	1	0	28	n 10900	1	0	40
B 2000	0	0	2	h 5000	+ 1	0	15	G 8000	0	- 1	27	m 11000	1	0	41
B 2100	+ 1	- 1	2	g 5100	+ 1	0	16	F 8100	1	0	28	m 11100	1	- 1	41
A 2200	+ 1	0	3	h 5200	0	- 1	15	E 8200	1	0	29	m 11200	0	0	41
u 2300	+ 1	0	4	g 5300	+ 1	0	16	E 8300	1	- 1	29	l 11300	1	0	42
A 2400	0	- 1	3	f 5400	+ 1	0	17	E 8400	0	0	29	l 11400	1	- 1	42
u 2500	+ 1	0	4	f 5500	+ 1	- 1	17	D 8500	1	0	30	k 11500	1	0	43
t 2600	+ 1	0	5	f 5600	0	0	17	D 8600	1	- 1	30	k 11600	0	0	43
t 2700	+ 1	- 1	5	e 5700	+ 1	0	18	C 8700	1	0	31	i 11700	1	0	44
t 2800	0	0	5	e 5800	+ 1	- 1	18	C 8800	0	0	31	i* 11800	1	- 1	44
s 2900	+ 1	0	6	d 5900	+ 1	0	19	C 8900	1	- 1	31	H 21600			86
s 3000	+ 1	- 1	6	d 6000	0	0	19	B 9000	1	0	32	p 31600			129
r 3100	+ 1	0	7	d 6100	+ 1	- 1	19	A 9100	1	0	33	a 41600			172
r 3200	+ 0	0	7	c 6200	+ 1	0	20	A 9200	0	0	33	t 51600			215
r 3300	+ 1	- 1	7	b 6300	+ 1	0	21	A* 9300	1	- 1	33	e 61600			258
q 3400	+ 1	0	8	c 6400	0	- 1	20	u 9400	1	0	34	C 71600			301
p 3500	+ 1	0	9	b 6500	+ 1	0	21	t 9500	1	0	35	i 81600			344
q 3600	0	- 1	8	a 6600	+ 1	0	22	u 9600	0	- 1	34	G 91600			387
p 3700	+ 1	0	9	P 6700	+ 1	0	23	t 9700	1	0	35	n 101600			430
n 3800	+ 1	0	10	a 6800	0	- 1	22	s 9800	1	0	36	c 201600			860
n 3900	+ 1	- 1	10	P 6900	1	0	23	s 9900	1	- 1	36	D 301600			1290
n 4000	0	0	10	N 7000	1	0	24	s 10000	0	0	36	C 301700			1291
m 4100	+ 1	0	11	N 7100	1	- 1	24	r 10100	1	0	37	C 301800			1291
l 4200	+ 1	0	12	N 7200	0	0	24	r 10200	1	- 1	37	B 301900			1292
l* 4300	+ 1	- 1	12	M 7300	1	0	25	q 10300	1	0	38	B 302000			1292
l 4400	0	0	12	M 7400	1	- 1	25	q 10400	0	0	38	B 302100			1292
k 4500	+ 1	0	13	H 7500	1	0	26	q 10500	1	- 1	38	A 302200			1293

&c., *ad infin.*

The Letters prefixed to the different centuries are called the *Index* Letters: there are thirty such different Letters, as will be more fully explained hereafter (Art. 129). The series of Letters recur in exactly the same order after 300,000 years, reckoned from 1600; in other words, from A. D. 301,600, as the Table shows.

84. As the Solar Equation causes *three descents* of the Golden Numbers every 400 years (beginning with A. D. 1700), while the Lunar Equation causes only *eight ascents* every 2500 years (beginning with A. D. 1800), it is plain, as before remarked, that there must be (with occasional exceptions) a continuous *descent*. The Solar Equation will cause 75 descents in 10,000 years, while the Lunar will cause but 32 ascents. Consequently, therefore, the ratio of descent to ascent is $\frac{75}{32}$, or about $\frac{9}{4}$. The period of the Solar Equation is 4 centuries; that of the Lunar is 25 centuries: the least common measure of both is, accordingly, 100 centuries, or 10,000 years. Hence, at the expiration of every 10,000 years, reckoning from A. D. 1700, *inclusive*, the Solar and Lunar Equations will recur in the same order as before. There will be the same *order* or variety of Index Letters, but not the same Letters themselves. Thus, for example, the Index Letters of the *two* centuries 1700 and 1800 are the *same*—viz., C; the Indices of the *three* centuries 1900, 2000, 2100, are also the same—viz., B; of 2200, A; of 2300, u; and of 2400 and 2500, also A and u. Similarly, the Index Letters of the *two* centuries 11,700, 11,800 have the *same* Index Letter, i; and the *three* centuries, 11,900, 12,000, 12,100, have the *same* Index Letter, h; then comes the Letter g for 12,200, and f for 12,300; in 12,400, 12,500, g and f are repeated: and so on throughout the second period of 10,000 years, the Index Letters follow each other in groups *similar* to, though not the same as, those of the first period. The same thing is true of the third period of 10,000 years, beginning with 21,700, and ending with 31,600. So also of the fourth, fifth, &c., periods: the Index *Letters* will be *different* for the corresponding intervals of 10,000 years, but the *order of their succession* will be the *same*, until we come to the *thirtieth* period, beginning A. D. 301,700, when the *Letters themselves* will recur in the same order as at first: viz., D will be the Index of 301,600, as it was of 1600; C will be the Index of 301,700 and 301,800, as it was of 1700 and 1800; B of 301,900, 302,000, 302,100, as it was of 1900, 2000, 2100; and so on *ad infin.*

Thus, after the expiration of 300,000 years, reckoned from A. D. 1700, exclusive, there will be, supposing the Gregorian Solar and Lunar Equations to be rigidly exact, a perfect ἀποκατάστασις, or restoration of the cycle of the Secular Equations. The Golden Numbers will also have passed through all the possible displacements, and resume their first position; and everything will go on again in the identical manner as after the year A. D. 1600.

85. We have seen (Art. 84) that the total *advance* of each Golden Number from the place which it occupied in the year A. D. 1600 (that is to say, from the place in which it was found after the reformation of the Calendar) will amount to *forty-three* places, after the expiration of 10,000 years, reckoned from that date (75 − 32 = 43), as the Table

shows, under A.D. 116,000. In other words, there will be a change of 43 Index Letters: that is, the whole cycle of 30 Letters, each change corresponding to the displacement of the Golden Numbers one day downwards, through the full Lunar month of 30 days, will be completed; and 13 displacements over will have taken place. Similarly, at the end of the second period, 21,600, there will be a total change of 86 Letters, or a *total* downward movement of the Golden Numbers through 86 places; the absolute amount of change, rejecting the two cycles of 30 Letters or displacements, being 26, reckoned from C of A. D. 1700. Similarly, at the end of the third period, A. D. 31,600, the total amount of descents will be 129 (43×3); or, omitting the 30's, 9; and, pursuing the same process, we shall find (as the Table shows) that, at the end of the 30th period—viz., A. D. 301,600—the *total* number of changes of Letters, or displacements of the Golden Numbers, will be 1290; which, divided by 30, leaves no remainder, showing that in the year 301,700, or after the lapse of 300,000 years (30 cycles of 10,000 each), the same Letter C will recur, and all the following ones in exactly the same order as before.

The number of displacements from the position of the Golden Numbers in 1700 may be easily expressed by the following formula:—Let n denote the number of periods of 10,000 years, reckoning from A. D. 1700 (Letter C); then the general expression for the number of Letters changed (or displacements of the Golden Numbers)

$$= \left(\frac{n \times 43}{30}\right)_r = \left(\frac{13n}{30}\right)_r.$$

If there is to be no displacement, n must be a multiple of 30: thus, the least number of cycles of 10,000 years that must elapse before a complete recurrence of Letters is 30.

To illustrate the use of this formula, let us take one or two examples.

Ex. 1. Required the number of Letters changed, or displacements of Golden Numbers, at the end of the 29th period of 10,000 years.

Evidently $$= \left(\frac{29 \times 13}{30}\right)_r = 17.$$

Ex. 2. Let $n = 30$. Then number of displacements

$$= \left(\frac{30 \times 13}{30}\right)_r = 0.$$

The series of remainders, or Letters changed, corresponding to the 30 cycles, are as follows:—

Cycles,	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Remainders,	13	26	9	22	5	18	1	14	27	10	23	6	19	2	15
Cycles,	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Remainders,	28	11	24	7	20	3	16	29	12	25	8	21	4	17	30

86. It appears, by inspection of the Equation Table (Art. 83), that in the year 8200—that is to say, in 6600, reckoning from A. D. 1600—the number of displacements of the Golden Numbers from their places after the reformation of the Calendar is 29; in other words, each Golden Number will have passed through the whole Paschal Lunation of 29 days in 6600 years: and in the year 8500, or 6900 years from A. D. 1600, all the Golden Numbers will have passed through 30 displacements, or (which is the same thing) will have come back to the *same place* in the month which they occupied immediately after the reformation of the Calendar; so that, in fact, their displacement then will be 0. They will remain stationary up to 8699; and in 8700 they will begin their second cycle of 30 displacements, which will be completed in A. D. 15,400. They will then continue stationary up to 15,699; and the third cycle of 30 displacements (another period of 6900 years, reckoning from 15,400) will commence in 15,700, and be completed in 22,300. This law, however, will not hold good for the next cycle of displacements, which will not be completed till A. D. 29,400; that is to say, 7100 from 22,300. The periods thenceforth will be 6900, 7000, and 7100, at certain intervals. But, without dwelling any longer on these details, it will be better to find the *General Rule for the number of displacements downwards in any century, reckoned from* 1600, *arising from the joint action of the Solar and Lunar Equations.*

87. Let us ascertain *the number of descents arising from the Solar Equation* for any century, reckoned from A. D. 1600. As the Solar Equation omits 3 intercalary days every 400 years, reckoned from 1600, these 3 omitted days will cause 3 descents of the Golden Numbers. Let σ denote the *centurial figures* of any century after A. D. 1600; and let $\odot$ denote the corresponding number of descents due to the Solar Equation. If, then, we divide $\sigma - 16$ by 4, and multiply the quotient by 3, we shall have the number of omitted days, or descents, in the complete groups of 4 contained in $\sigma - 16$; the remainder, if any, which of course must be less than 4, will be *Common* centurial years, and therefore must be added to the above product in order to find the whole number of omitted days. Hence we have

$$\odot = \left(\frac{\sigma - 16}{4}\right)_w \times 3 + \left(\frac{\sigma - 16}{4}\right)_r. \qquad (1)$$

This may be otherwise expressed. The total number of omitted days is less than the total number of centuries $(\sigma - 16)$ by $\frac{1}{4}$ of the latter: hence

$$\odot = \sigma - 16 - \left(\frac{\sigma - 16}{4}\right)_w. \qquad (2)$$

The former is Clavius' formula (ch. xii. § 9); the latter is Delambre's (1). I shall adopt the latter. Accordingly, we have the following Rule for finding *the number of omitted days due to the Solar Equation* for any century A. D.:—

Subtract 16 *from the number of the century; and from the difference subtract its fourth part, omitting fractions: the result will be the required number* (2).

Ex. 1. Required the number of descents due to the Solar Equation for the year 8000 A. D.

Here $\sigma = 80$, and therefore, by formula (2),

$$\odot = 64 - \left(\frac{64}{4}\right)_w = 64 - 16 = 48.$$

Ex. 2. Required the number of descents due to the Solar Equation for 8500 A. D.

Here

$$\odot = 69 - \left(\frac{69}{4}\right)_w = 69 - 17 = 52\ (^3).$$

(1). They are easily reconcileable algebraically. For

$$\sigma - 16 = 4\left(\frac{\sigma - 16}{4}\right)_w + \left(\frac{\sigma - 16}{4}\right)_r.$$

If we substitute this value for $\sigma - 16$ in (2), we get (1).

(2). As we shall have frequently to do with such quantities as

$$\left(\frac{m}{4}\right)_w \text{ and } \left(\frac{m}{4}\right)_r,$$

it may be as well to give one or two of the elementary rules for the combination of

$$\left(\frac{m}{4}\right)_w, \left(\frac{n}{4}\right)_w; \text{ and } \left(\frac{m}{4}\right)_r, \left(\frac{n}{4}\right)_r.$$

Let $m = 4p + q$, so that

$$p = \left(\frac{m}{4}\right)_w,\ q = \left(\frac{m}{4}\right)_r;$$

$n = 4p' + q'$, so that

$$p' = \left(\frac{n}{4}\right)_w,\ q' = \left(\frac{n}{4}\right)_r.$$

Then (*a*)

$$\left(\frac{m}{4}\right)_r + \left(\frac{n}{4}\right)_r = q + q',$$

$$\left(\frac{m+n}{4}\right)_r = \left(\frac{4\overline{p+p'} + \overline{q+q'}}{4}\right)_r = \left(\frac{q+q'}{4}\right)_r.$$

So $$\left(\frac{m+n}{4}\right)_r = \left(\frac{m}{4}\right)_r + \left(\frac{n}{4}\right)_r, \quad \text{if } q + q' < 4,$$

$$= \left(\frac{m}{4}\right)_r + \left(\frac{n}{4}\right)_r - 4, \text{ if } q + q' \begin{smallmatrix}=\\>\end{smallmatrix} 4.$$

(*b*) $$\left(\frac{m}{4}\right)_r - \left(\frac{n}{4}\right)_r = q - q' \text{ (} m \text{ supposed greater than } n\text{)};$$

$$\left(\frac{m-n}{4}\right)_r = \left(\frac{4\,(p-p') + (q-q')}{4}\right)_r.$$

(1) If $q - q'$ be zero or positive, the last-written bracket $= q - q'$.

(2) If $q - q'$ be negative, the bracket $= \left(\frac{4\,\overline{p-p'} + 3\,(q'-q)}{4}\right)_r$

$$= \left(\frac{3\,(q'-q)}{4}\right)_r.$$

This bracket = 1, 2, 3, according as $q' - q = 3, 2, 1$, respectively. That is, it $= 4 - (q' - q)$.

So $$\left(\frac{m-n}{4}\right)_r = \left(\frac{m}{4}\right)_r - \left(\frac{n}{4}\right)_r, \text{ when } q - q' \text{ is zero or positive,}$$

$$= 4 + \left(\frac{m}{4}\right)_r - \left(\frac{n}{4}\right)_r, \text{ when } q - q' \text{ is negative.}$$

(*c*) $$\left(\frac{m}{4}\right)_w + \left(\frac{n}{4}\right)_w = p + p'.$$

$$\left(\frac{m+n}{4}\right)_w = \left(\frac{4\,\overline{p+p'} + \overline{q+q'}}{4}\right)_w = p + p', \text{ if } q + q' < 4$$

$$= p + p' + 1, \text{ if } q + q' \begin{smallmatrix}=\\>\end{smallmatrix} 4.$$

So $$\left(\frac{m+n}{4}\right)_w = \left(\frac{m}{4}\right)_w + \left(\frac{n}{4}\right)_w, \text{ if } q + q' < 4,$$

$$= \left(\frac{m}{4}\right)_w + \left(\frac{n}{4}\right)_w + 1, \text{ if } q + q' \begin{smallmatrix}=\\>\end{smallmatrix} 4.$$

(*d*) $$\left(\frac{m}{4}\right)_w - \left(\frac{n}{4}\right)_w = p - p' \text{—} (m > n).$$

$$\left(\frac{m-n}{4}\right)_w = \left(\frac{4\,\overline{p-p'} + q - q'}{4}\right)_w = \left(\frac{m}{4}\right)_w - \left(\frac{n}{4}\right)_w, \text{ if } q - q' \text{ be positive or zero,}$$

$$= \left(\frac{m}{4}\right)_w - \left(\frac{n}{4}\right)_w - 1, \text{ if } q - q' \text{ be negative.}$$

These theorems may of course be extended generally as follows:—
Suppose

$$\left.\begin{aligned} m &= \kappa \cdot p + q \\ n &= \kappa \cdot p' + q', \end{aligned}\right\} \text{ and } m > n;$$

then

(a′) $$\left(\frac{m+n}{\kappa}\right)_r = \left(\frac{m}{\kappa}\right)_r + \left(\frac{n}{\kappa}\right)_r, \text{ if } q+q' < \kappa;$$

$$= \left(\frac{m}{\kappa}\right)_r + \left(\frac{n}{\kappa}\right)_r - \kappa, \text{ if } q+q' \geqq \kappa.$$

(b′) $$\left(\frac{m-n}{\kappa}\right)_r = \left(\frac{m}{\kappa}\right)_r - \left(\frac{n}{\kappa}\right)_r, \text{ if } q-q' \text{ be zero or positive;}$$

$$= \kappa + \left(\frac{m}{\kappa}\right)_r - \left(\frac{n}{\kappa}\right)_r, \text{ if } q-q' \text{ be negative.}$$

(c′) $$\left(\frac{m+n}{\kappa}\right)_w = \left(\frac{m}{\kappa}\right)_w + \left(\frac{n}{\kappa}\right)_w, \text{ if } q+q' < \kappa,$$

$$= \left(\frac{m}{\kappa}\right)_w + \left(\frac{n}{\kappa}\right)_w + 1, \text{ if } q+q' \geqq \kappa.$$

(d′) $$\left(\frac{m-n}{\kappa}\right)_w = \left(\frac{m}{\kappa}\right)_w - \left(\frac{n}{\kappa}\right)_w, \text{ if } q-q' \text{ be zero or positive,}$$

$$= \left(\frac{m}{\kappa}\right)_w - \left(\frac{n}{\kappa}\right)_w - 1, \text{ if } q-q' \text{ be negative.}$$

There is an important particular case of this last formula, which deserves special notice, viz., the case where $m = \text{mult. } \kappa$: *i. e.*, where $q = 0$.

There are then two cases to consider: (1) when q' is not $= 0$; (2) when $q' = 0$.

(1). q' not $= 0$. We have, since $q - q'$ is negative—

$$\left(\frac{m}{\kappa}\right)_w - \left(\frac{n}{\kappa}\right)_w = \left(\frac{m-n}{\kappa}\right)_w + 1 = \left(\frac{m-n-1}{\kappa}\right)_w + 1,$$

since $m - n$ is not a multiple of κ.

(2). $q' = 0$. We have

$$\left(\frac{m}{\kappa}\right)_w - \left(\frac{n}{\kappa}\right)_w = \left(\frac{m-n}{\kappa}\right)_w = \left(\frac{m-n-1}{\kappa}\right)_w + 1,$$

since $m - n$ is a multiple of κ.

Thus *both* cases are included in the formula

$$\left(\frac{m}{\kappa}\right)_w - \left(\frac{n}{\kappa}\right)_w = \left(\frac{m-n-1}{\kappa}\right)_w + 1. \qquad (e)$$

Another theorem easily deducible is the following. If m be any number, then

$$\left(\frac{3(m+1)}{4}\right)_w = m - \left(\frac{m}{4}\right)_w. \qquad (e)$$

For, with the same notation as before,

$$\left(\frac{3(m+1)}{4}\right)_w = \left(\frac{12p+3q+3}{4}\right)_w = 3p + \left(\frac{3q+3}{4}\right)_w, \text{ by means of } (c);$$

$$= 3p + \left(\frac{4q+3-q}{4}\right)_w = 3p + q, \text{ since } 3-q \text{ is} < 4;$$

$$= m - p = m - \left(\frac{m}{4}\right)_w.$$

By means of this theorem we may convert equation (2) of Art. 87 into the form—

$$\odot = \left(\frac{3(\sigma - 15)}{4}\right)_w. \qquad (b)$$

(3). It is sometimes useful to solve the converse problem: viz., being given the amount of the Solar Equation, to find the century σ.

Taking equation (2) of Art. 87—

$$\odot = \sigma - 16 - \left(\frac{\sigma - 16}{4}\right)_w$$

$$= \sigma - 12 - \left(\frac{\sigma}{4}\right)_w;$$

but $$\sigma - \left(\frac{\sigma}{4}\right)_w = \left(\frac{3(\sigma+1)}{4}\right)_w, \text{ by equation } (e) \text{ of Note 2.}$$

Therefore $$\left(\frac{3(\sigma+1)}{4}\right)_w = \odot + 12 = a \text{ say.}$$

To solve this equation, I will assume $\sigma + 1 = 4p + q$: then

$$a = \left(\frac{3(4p+q)}{4}\right)_w = 3p + \left(\frac{3q}{4}\right)_w$$

$$= 3p + \left(\frac{4q - q}{4}\right)_w = 3p + q - 1,$$

by equation (d) of Note 2, and observing that $\left(\frac{q}{4}\right)_w = 0$,

but $= 3p$, if $q = 0$.

Hence $$3p + q = a + 1,$$

or else $3p = a$.

Taking the first supposition,

$$4p + q = \frac{4(a+1)}{3} - \frac{q}{3}$$

$$= \frac{4a}{3} + 1, \text{ if } q = 1,$$

$$= \frac{4a}{3} + \tfrac{2}{3}, \text{ if } q = 2,$$

$$= \frac{4a}{3} + \tfrac{1}{3}, \text{ if } q = 3,$$

or, in general, $$\sigma + 1 = \left(\frac{4a}{3}\right)_w + 1,$$

$$\sigma = \left(\frac{4a}{3}\right)_w.$$

Taking the second supposition, $\sigma + 1 = \frac{4a}{3}$;

which can only be true when $\frac{a}{3}$ is an integer.

In other words: whenever a is a multiple of 3, σ has two values, viz., $\frac{4a}{3}$ and $\frac{4a}{3} - 1$: in other cases σ has the single value $\left(\frac{4a}{3}\right)_w$.

Ex. 1. Given $\odot = 7, \quad a = 19, \quad \frac{4a}{3} = 25\frac{1}{3}, \therefore \sigma = 25.$

Ex. 2. Given $\odot = 11, \quad a = 27, \quad \frac{4a}{3} = 36,$

and, therefore, $\sigma = 35$ or 36.

88. To find the number of *ascents* due to the *Lunar Equation* for any century A. D.

The first Lunar Equation, or ascent, took place A. D. 1800. After that date, and reckoning from it, there are 8 Equations, or ascents, in every 2500 years: namely, one for 7 consecutive periods of 300 years, up to A. D. 3900, inclusive; and the eighth at the end of the next 400 years, or A. D. 4300. The next 2500 years, distributed in the same way, ends A. D. 6800; the third period, A. D. 9300; and so on. Hence, it is easy to find a formula for the number of ascents, which may be denoted by ☽, for any century, σ, reckoned from 1800.

Divide $\sigma - 18$ by 25. Let the quotient be Q, and the remainder R. Multiply Q by 8, and the product will be the number of ascents due to the number of complete cycles of 25 centuries contained in $\sigma - 18$. As R is less than 25, divide it by 3, and we shall have the number of groups of 3 centuries, or (which is the same thing) the number of ascents contained in R, remembering that $\frac{R}{3}$ cannot exceed 7, by the conditions of the problem—viz., 8 ascents in twenty-*five* centuries: so that, if R should be 24, the quotient must be taken as 7, not 8. Hence, adding 1 for the Lunar Equation in 1800, we get the total number of ascents (☽) from 1600 to σ—

$$☽ = 1 + 8 \times \left(\frac{\sigma - 18}{25}\right)_w + \left(\frac{\sigma - 18}{25}\right)_r \times \tfrac{1}{3}, \qquad (3)$$

where $\frac{R}{3}$ cannot be greater than 7.

This is Clavius' formula (cap. xii. § 9). His Rule, therefore, for finding the total amount of displacement *upwards*, due to the Lunar Equation, from the date of the Reformation of the Calendar, is this:—"*Subtract* 18 *from the number of the century; take the quotient of the difference divided by* 25, *and multiply this quotient by* 8. *If there be a remainder, divide it by* 3, *and add the result, which must be less than* 8, *to the number already found: and to the* SUM *of both add* 1. *The resulting sum will be the number required.*"

It is obvious that the *second* term of (3) will not come into use until $\left(\frac{\sigma - 18}{25}\right)$ becomes a *whole number;* that is to say, until A. D. 4300, when $\sigma - 18$ $(= 43 - 18 = 25)$ becomes

= 1, and ∴ the second term = 8. Up to that date the Lunar Equation is given by the *third* term; that is to say, $☽ = 1 + \left(\frac{\sigma - 18}{25}\right)_r \times \frac{1}{3}$, where $\frac{R}{3}$ must be less than 8.

We will now give some examples to illustrate the use of the above formula (3).

Ex. 1. Find ☽ for A. D. 2100.

The second term will not have come into use; and the third term

$$= \tfrac{1}{3}\left(\frac{21 - 18}{25}\right)_r = \tfrac{3}{3} = 1.$$

Hence, we get $$☽ = 1 + 1 = 2,$$
the number sought.

Ex. 2. Find ☽ for A. D. 4200.

The second term must be omitted. The third term

$$= \left(\frac{42 - 18}{25}\right)_r = 24,$$

which, divided by 3, would give 8; but (as above explained) we must take 7. Hence,

$$☽ = 1 + 7 = 8,$$

the required number.

Ex. 3. Find ☽ for A. D. 8500.

Here the second term

$$= \left(\frac{85 - 18}{25}\right)_w \times 8 = 2 \times 8 = 16;$$

the third term $$= \left(\frac{67}{25}\right)_r \times \tfrac{1}{3} = \frac{17}{3} = 5.$$

Hence $$☽ = 1 + 16 + 5 = 22.$$

Ex. 4. Find ☽ for A. D. 11,600: *i. e.*, 10,000 years after 1600.

Here, the second term

$$= \left(\frac{116 - 18}{25}\right)_w \times 8 = \left(\frac{98}{25}\right)_w \times 8 = 3 \times 8 = 24.$$

The third term $$= \left(\frac{98}{25}\right)_r \times \tfrac{1}{3} = \left(\frac{23}{3}\right) = 7:$$

hence, $$☽ = 1 + 24 + 7 = 32.$$

Delambre has given (Astron. Mod., i. p. 9) the following elegant expression for the total amount of the Lunar Equation in any century, σ, of the Christian Era, after the correction of the Calendar—

$$☽ = \left(\frac{\sigma - 15 - a}{3}\right)_w, \tag{4}$$

where $$a = \left(\frac{\sigma - 17}{25}\right)_w.$$

This may be proved thus. If the Lunar Equation were one day every 300 years, reckoning from 1800, its total amount for any century A. D. (σ) would be $\frac{1}{3}$rd of the number of centuries elapsed between 1800 and σ, omitting fractions: that is, $\left(\frac{\sigma - 18}{3}\right)_w$; and this formula actually holds for the 23 centuries from 1800 to 4100; but when we come to A. D. 4200 (the twenty-fourth century), the correction is DEFERRED till the next century, 4300 (the twenty-fifth century, reckoned from 1800); or, which is the same thing, century 4200 is dropped out of the reckoning, and the *four* centuries from 3900 to 4300 are dealt with as if they were only *three*. In the same way, the correction in A. D. 6700 (the next twenty-fourth century, reckoning from 43) is deferred to A. D. 6800, and the *four* centuries from 6400 to 6800 are again dealt with as if they were only *three*. Similarly, the correction in A. D. 9200 is deferred to 9300, and so on continually. In this way, by dropping out the centuries 42, 67, 92, &c., with their corrections, we in effect render the total number of corrections equal to $\frac{1}{3}$rd of the number of centuries ($\sigma - 18$) thus reduced. But

$$42 = 17 + 25;\ 67 = 17 + 2 \cdot 25;\ 92 = 17 + 3 \cdot 25;\ \&c.;$$

so that $$\frac{42 - 17}{25} = 1;\ \left(\frac{67 - 17}{25}\right)_w = 2:$$

consequently, the number of centuries dropped out, or the number of deferred corrections, for any century (σ) may be represented by $\left(\frac{\sigma - 17}{25}\right)_w$. This is the correction to be applied to $\left(\frac{\sigma - 18}{3}\right)_w$, on account of the deferring of every *eighth* equation to the 400th year. Consequently, calling this $\left(\frac{\sigma - 17}{25}\right)_w$, a, we get the total Lunar Equation for any century—

$$☽ = \left(\frac{\sigma - 18 - a}{3}\right)_w;$$

adding 1 to this, for the Lunar Equation in 1800, we find

$$☽ = \left(\frac{\sigma - 15 - a}{3}\right)_w \text{(1)}.$$

It is obvious, from the above formula (4), that the value of a will be 0 till $\sigma - 17 = 25$, or $\sigma = 42$; therefore, until the year 4200, a may be neglected in the computation of the Lunar Equation. Had the proemptosis been taken, as it ought more correctly to have been, at one day in 308 years (Art. 74), instead of $312\frac{1}{2}$ years, the Lunar Equation would have taken place only twelve times in 3700 years, or eleven times successively at the end of 300 years, and the twelfth at the end of the following 400 years. On this supposition, it is easy to show, in the same manner as before, that the value of a would be $\left(\frac{\sigma - 17}{37}\right)_w$, and, therefore, a would be 0 until $\sigma - 17 = 37$, or until A. D. 5400.

(1). The following proof is given by Delambre :—

Let x be the number of centuries from 1800 to the century of the Christian Era for which the total amount of the Lunar Equation is sought. The conditions of the problem are: 1°. There are to be 8 corrections in every 2500 years, reckoning from 1800; and, 2°. that 7 of these corrections are to be made consecutively at intervals of 300 years, the eighth being deferred to the 400th year.

By the first condition, the total number of corrections for x centuries is obviously $\left(\frac{8x}{25}\right)_w$. But

$$\left(\frac{8x}{25}\right)_w = \left(\frac{25x}{75} - \frac{x}{75}\right)_w = \left(\frac{x - \left(\frac{x}{25}\right)_w}{3}\right)_w.$$

By the second condition $\left(\frac{x}{25}\right)_w$ must be taken as 1 $\left(= \frac{x+1}{25}\right)$ when $x = 24$, because $\left(\frac{24-0}{3}\right)_w = 8$, whereas 8 cannot come till $x = 25$. Similarly, when $x = 25 + 24$, $\left(\frac{x}{25}\right)_w$ must be taken as 2; and generally when $x = m \, . \, 25 + 24$, $\left(\frac{x}{25}\right)_w$ must be taken as $m + 1$. Instead, therefore, of $\left(\frac{x}{25}\right)_w$, we must write $\left(\frac{x+1}{25}\right)_w$, and the general expression for the Lunar Equation for x centuries contained between 1800 and the given century, σ, is

$$\left(\frac{x - \left(\frac{x+1}{25}\right)_w}{3}\right)_w = \left(\frac{\sigma - 18 - \left(\frac{\sigma - 17}{25}\right)_w}{3}\right)_w,$$

and adding 1 for the Lunar Equation in 1800, we get finally,

$$☽ = \left(\frac{\sigma - 15 - \left(\frac{\sigma - 17}{25}\right)_w}{3}\right)_w.$$

Delambre (Astr. Mod. i., p. 69) gives another expression for the Lunar Equation, due to the Abbé Titte[1], Götting., 1816. It is

$$\mathfrak{D}' = \left(\frac{8\sigma + 13}{25}\right)_w + 2.$$

It may be deduced from equation (4) of Art. 88, at once, thus:

$$\mathfrak{D} = \left(\frac{\left(\frac{25\,(\sigma - 15)}{25}\right)_w - \left(\frac{\sigma - 17}{25}\right)_w}{3}\right)_w.$$

This is a case in which equation (e') of Art. 87, Note 2, applies.
We have then

$$\mathfrak{D} = \left\{\frac{\left(\frac{25\sigma - 375 - \sigma + 17 - 1}{25}\right)_w + 1}{3}\right\}_w$$

$$= \left\{\frac{\left(\frac{24\sigma - 334}{25}\right)_w}{3}\right\}_w$$

$$= \left(\frac{8\sigma - 112}{25}\right)_w$$

$$= \left(\frac{8\sigma + 13}{25}\right) - 5,$$

showing that $\mathfrak{D} = \mathfrak{D}' - 7.$

89. We can now find the *combined* effect of the Solar and Lunar Equations for an century A. D. (σ) after 1600.

By formula (2), Art. 87, the total Solar Equation, or number of *descents*, is,

$$\odot = \sigma - 16 - \left(\frac{\sigma - 16}{4}\right)_w.$$

By formula (4), Art. 88, the total Lunar Equation, or number of *ascents*, is,

$$\mathfrak{D} = \left(\frac{\sigma - 15 - a}{3}\right)_w.$$

But as the number of descents exceeds the number of ascents, the *actual number of displacements* downwards due to *both causes* combined is

$$\odot - \mathfrak{D} = \sigma - 16 - \left(\frac{\sigma - 16}{4}\right)_w - \left(\frac{\sigma - 15 - a}{3}\right)_w. \qquad (1)$$

Ex. 1. Find the total displacement downwards for A. D. 2000.

Here $\odot = 3$; and $\mathfrak{D} = 1$. Hence the required displacement = 2.

Ex. 2. Find the total displacement downwards for A. D. 6700.

Here $\odot = 39$; and ☽ $= 16$; because $67 - 15 = 52$, and $a = 2$; consequently,

$$\left(\frac{52 - 2}{3}\right)_w = 16.$$

Hence, $$\odot - ☽ = 39 - 16 = 13.$$

Ex. 3. Find total displacement downwards for A. D. 8500.

Here $\odot = 52$; and ☽ $= 22$; therefore, $\odot -$ ☽ $= 52 - 22 = 30$, or $0\ \odot$.

When the number of displacements exceeds 30, we must omit 30, and take the remainder. In other words, formula (1), above, must be written in its most general form,

$$\odot - ☽ = \left\{\frac{\sigma - 16 - \left(\frac{\sigma - 16}{4}\right)_w - \left(\frac{\sigma - 15 - a}{3}\right)_w}{30}\right\}_r. \qquad (2)$$

Ex. 4. Find the total displacement downwards for A. D. 11,600, or 10,000 after the reformation of the Calendar (1600).

Here we have $\odot = 75$; ☽ $= 32$; therefore,

$$\odot - ☽ = \left(\frac{75 - 32}{30}\right)_r = 13.$$

90. The Equation Table given by Clavius (cap. xi., pp. 134, *sq.*), and of which I have extracted a portion in Art. 83, was originally drawn up by Lilius. Clavius, who calculated the mean New Moons for many thousands of years, observed that the *Calendar* New Moons found by means of Lilius' Equation Table agreed very nearly with the mean New Moons as far as A. D. 8100; but that for the subsequent centuries the agreement did not hold. He, therefore, drew up a *corrected* Equation Table, beginning with A. D. 8200, and extending to A. D. 28,400 (1). This Table is found in cap. xii., p. 183. In this corrected Table the Index Letter, in passing from 8100 to 8200, changes from F to D (two places), instead of from F to E (one place), as in Lilius' Table. The result of this interruption of the regular order is that the cycle of 300,000 years (Art. 84) ceases to hold good, and the ἀποκατάστασις effected by Lilius' Table does not take place.

(1). The reason why he did not extend the Table beyond (!) 28,400 was, that probably after that year the Equinox will cease to be fixed to March 21, because the suppression of 3 Bissextile years in every 400, reckoned from 1600, instead of in every 402 years, as it should be (Clav., p. 86), would give rise to an error of one day in 26,800 years;

$$3\left(\frac{1}{400}-\frac{1}{402}\right)=\frac{6}{160800}=\frac{1}{26800}.$$

(2). The reason why Lilius' Table ceases to be applicable after 8100, and why it is necessary to descend *two* Letters instead of *one*, in passing from 8100 to 8200, is given by Clavius, cap. xii., 10. It depends on the *two* Epacts 25, 24, being affixed to the *same* day, February 25.

91. Having now discussed the effects upon the position of the *Golden Numbers*, produced by the Gregorian reformation of the Calendar, we have next to inquire what changes it made in the Rules for determining the *Sunday Letters*.

I have already shown (Arts. 31, 32, 33) how these Letters (Old Style) may be found by a Table and by arithmetical formulæ. I shall now proceed to give the corresponding Table and formulæ for the New Style.

The effect of omitting the 10 nominal days between Oct. 4th and Oct. 15th, was, of course, to drop the 10 Calendar Letters corresponding to those 10 days: in other words, to diminish by 10 the number of *regressions* (Art. 32) that would have taken place had the Julian reckoning remained unaltered. But as 10 = 7 + 3, and as the 7 is dropped in the division by 7, the diminution of the number of regressions was 3. Now we have seen that the number of the Julian Sunday Letter, or the number of regressions, calculated by reference to the Prayer Book scale,

A	G	F	E	D	C	B
0	1	2	3	4	5	6,

for any year x A.D. is—

$$L=\left(\frac{x+\left(\frac{x}{4}\right)_w+5}{7}\right)_r;$$

from which, if we deduct 3 before dividing by 7, we get

$$L=\left(\frac{x+\left(\frac{x}{4}\right)_w+2}{7}\right)_r. \qquad (1)$$

This is the expression for the *Sunday Letter after the correction* in 1582; and as no further omission of a day, or Solar Equation, takes place until 1700, this expression will give the Gregorian, or New Style, Sunday Letter from Oct. 15, 1582, to the end of the year 1699.

Thus the following Rule, for the period just defined:—

"*To the number of the year add its fourth part, omitting fractions,* and to the result add 2; divide the sum thus obtained by 7: if there be no remainder the Sunday Letter is A: if there be a remainder, the Letter in the above scale corresponding to that remainder, will be the Letter required."

Ex. 1. Find Sunday Letter for 1582, *after* the correction.

Here we have $\left(\frac{1582 + 395 + 2}{7}\right)_r = \left(\frac{1979}{7}\right)_r = 5 =$ (by the scale) C.

The Letter before the correction was G, 3 regressions further on.

Ex. 2. Find Sunday Letters for 1632 (Leap-year).

Here we have $$\left(\frac{1632 + 408 + 2}{7}\right)_r = 5 = C.$$

But as this was Leap-year, C is the *second* Letter (Art. 32); and, therefore, the required Sunday Letters are D C.

92. In 1700 the *Solar Equation* takes place, which causes the omission of another day: that is to say, we must deduct another unit from formula (1) of Art. 91; which gives us for the century 1700–1799, inclusive,

$$L = \left(\frac{x + \left(\frac{x}{4}\right)_w + 1}{7}\right)_r.$$

This is the Rule given in the Prayer Book, viz.: "*Add to the year of our Lord its fourth part, omitting fractions; and also the number* 1: *divide the sum by* 7; *and if there is no remainder, then* A *is the Sunday Letter: but if any number remaineth, then the Letter standing against that number, in the same annexed Table, is the Sunday Letter.*"

0	A
1	G
2	F
3	E
4	D
5	C
6	B

Ex. 1. Find Sunday Letter for A. D. 1734.

Here we have

$$\left(\frac{1734 + 433 + 1}{7}\right)_r = \left(\frac{2168}{7}\right)_r = 5 = \text{C}.$$

Ex. 2. Find the Sunday Letters for 1748 (Leap-year).

We have

$$\left(\frac{1748 + 437 + 1}{7}\right)_r = \left(\frac{2186}{7}\right)_r = 2 = \text{F}. \quad \text{Consequently the Letters are G F.}$$

93. In *general*, the number of *Letters* omitted will be 10, *plus* the number of days dropped on account of the secular Solar Equation, which we have already seen (Art. 87) to be $= \sigma - 16 - \left(\frac{\sigma - 16}{4}\right)_w$. Hence, subtracting this from the formula (1), Art. 91, which gives number of the Sunday Letter after the omission of the 10 days, and before the secular Solar Equation begins, we get as the general expression for L for any year A. D. x

$$L = \left(\frac{x + \left(\frac{x}{4}\right)_w + 2 - (\sigma - 16) + \left(\frac{\sigma - 16}{4}\right)_w}{7}\right)_r \tag{2}$$

$$= \left(\frac{x + \left(\frac{x}{4}\right)_w - \sigma + \left(\frac{\sigma}{4}\right)_w}{7}\right)_r \tag{3}$$

$$\left.\begin{aligned} &= \left(\frac{x + \left(\frac{x}{4}\right)_w}{7}\right)_r - \left(\frac{\sigma - \left(\frac{\sigma}{4}\right)_w}{7}\right)_r \text{ if the first remainder exceed the second by equation } (b') \text{ of Art. 87, Note 2;} \\ &\text{or } 7 + \left(\frac{x + \left(\frac{x}{4}\right)_w}{7}\right)_r - \left(\frac{\sigma - \left(\frac{\sigma}{4}\right)_w}{7}\right)_r \text{ if the second remainder exceed the first.} \end{aligned}\right\} \tag{4}$$

Both these last equations are included in the general expression

$$L = \left(\frac{7 + x_1 - \sigma_1}{7}\right)_r, \tag{5}$$

where

$$x_1 = \left(\frac{x + \left(\frac{x}{4}\right)_w}{7}\right)_r, \quad \sigma_1 = \left(\frac{\sigma - \left(\frac{\sigma}{4}\right)_w}{7}\right)_r.$$

Let us apply these expressions to some examples.

Ex. 1. Sunday Letter for 1875 A.D.

Here $x + \left(\frac{x}{4}\right)_w = 2343$; so $x_1 = 5$; $\sigma - \left(\frac{\sigma}{4}\right)_w = 14$; $\sigma_1 = 0$.

Thus $L = 5 = C$, by the scale, Art. 91.

For the 19th century, $\sigma_1 = 0$, and $L = \left(\frac{x + \left(\frac{x}{4}\right)_w}{7}\right)_r$; which is the Rule given in the Prayer Book for finding the Sunday Letter from 1800 to 1899.

This Rule holds good, whenever $\sigma - \left(\frac{\sigma}{4}\right)_w$ is a multiple of 7.

94. We have seen (Art. 33) that in the Julian reckoning the Sunday Letter L may also be found by means of the formula (2), Art. 33, Note.

$$L = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w - 3}{7}\right)_r, \tag{1}$$

the scale to which it is referred being

A	B	C	D	E	F	G
1	2	3	4	5	6	7,

where the numbers correspond to the natural order of the seven Calendar Letters.

To adapt this formula (3) to the Gregorian Calendar, we have only to deduct from the quantities within the brackets the number 10, due to the omission of the 10 days in 1582, and the Solar Equation $\sigma - 16 - \left(\frac{\sigma - 16}{4}\right)_w$: which will give

$$L = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w - 3 - 10 - (\sigma - 16) + \left(\frac{\sigma - 16}{4}\right)_w}{7}\right)_r.$$

Add 14 to the numerator of the quantity within the bracket, and we get

$$L = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w + 1 - (\sigma - 16) + \left(\frac{\sigma - 16}{4}\right)_w}{7}\right)_r \tag{2}$$

$$= 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w + 6 - \sigma + \left(\frac{\sigma}{4}\right)_w}{7}\right)_r \tag{3}$$

or, if the last numerator be called R, $L = 7 - \left(\frac{R}{7}\right)_r$, which is Delambre's expression for L in the natural scale above given.

Ex. Find L for 1875.

Here $$x + \left(\frac{x}{4}\right)_w + 1 = 2344, \text{ and } \sigma - 16 = 2.$$

So $$L = 7 - \left(\frac{2342}{7}\right)_r = 7 - 4 = 3 = \text{C}.$$

It is to be observed that the term $\left(\frac{\sigma - 16}{4}\right)_w$ does not come into use till A.D. 2000.

Returning to formula (2), Art. 93, we shall see that, if we put $\left(\dfrac{x + \left(\frac{x}{4}\right)_w}{7}\right)_r = \bar{x}$, we have the following values for L for the first seven centurial years after the Gregorian correction :—

From	1583	to	1699	(inclusive),	$L = \bar{x} + 2.$
From	1700	to	1799	,,	$L = \bar{x} + 1$
From	1800	to	1899	,,	$L = \bar{x} + 0$
From	1900	to	2099	,,	$L = \bar{x} + 6$
From	2100	to	2199	,,	$L = \bar{x} + 5$
From	2200	to	2299	,,	$L = \bar{x} + 4$
From	2300	to	2499	,,	$L = \bar{x} + 3$
From	2500	to	2599	,,	$L = \bar{x} + 2$
From	2600	to	2699	,,	$L = \bar{x} + 1$
From	2700	to	2899	,,	$L = \bar{x} + 0;$

and so on for each successive series of nine centuries. The numbers added to $\bar{x}$, after passing through zero, go to the other end of the scale, 6.

As the number of *regressions* of the Julian Sunday Letter is diminished in the Gregorian reckoning by 10; and further by $\odot$, the Solar Equation; if L' be the number of the Julian Sunday Letter for any year A.D., x, and L be the Gregorian Sunday Letter for the same year, we have

$$L' - L = \left(\frac{10 + \odot}{7}\right)_r, \text{ in } \textit{retrograde} \text{ order of Letters;} \qquad (4)$$

or, $$L - L' = \left(\frac{10 + \odot}{7}\right)_r, \text{ in } \textit{direct} \text{ order of Letters.} \qquad (5)$$

The condition of coincidence between L and L' is obviously

$$10 + \odot = \text{mult. } (7),$$

or,

$$\odot = 4, 11, 18, \&c.$$

Ex. 1. Required the difference between the Old and New Style Letter for A.D. 1600.

Here $\odot = 0$, and therefore $L' - L = \left(\frac{10}{7}\right)_r = 3.$

But by the expression (Art. 91) $L' = \left(\frac{x + \left(\frac{x}{4}\right)_w + 5}{7}\right)_r = 3 = $ F E, since it is Leap-year.

So $L = $ B A.

Ex. 2. Find $L' - L$ for A.D. 2603.

Here $\odot = 8$, and $L' - L = \left(\frac{10 + 8}{7}\right)_r = 4.$

Also $L' = \left(\frac{3258}{7}\right)_r = 3 = $ E; and therefore L, which is four places behind it, is B.

95. The above formulæ are quite sufficient to find the Gregorian Sunday Letter for any year A.D. But it is convenient to have a Table which will show it at once on inspection, as in the case of the Julian Sunday Letter (Arts. 30 and 31).

To construct this Table, it is only necessary to bear in mind that, as the Gregorian suppression of certain centurial Leap-years (Art. 77) interrupts the regular order of the Letters in the Solar Cycle of 28 years, this regular order will hold good for only a century. The intercalary period in the Gregorian reckoning is 400 years, in which there are 97 Leap-years; instead of four years with one Leap-year, as in the Julian. Hence it follows that after the lapse of any period of 2800 years (7 × 400) the days of the week must fall on the same days of the year as they did in the preceding period; just as in the case of the 28-year cycle in the Julian reckoning. But the recurrent coincidence will actually take place in a much shorter period than 2800 years, namely in 400 years: because, since the Sunday Letter retrogrades 5 places every four years, it will retrograde 500 places in 400 years; which, dividing by 7, is in fact equivalent to 3; $\left(\frac{500}{7}\right)_r = 3$. But the Gregorian Rule *suppresses three* intercalations in the same time. Hence, the same coincidence of week-days and month-days recurs every 400 years. Accordingly, a Table of Sunday Letters for 400 years will show the Sunday Letters in the Gregorian reckoning, for ever. Strictly speaking, this Table begins with October

15th, A.D. 1582. We divide the Table into four columns, containing the units and tens in 100 years; and a second set of four columns containing at the top the *centurial* years, beginning with 1600, in groups of four: 16, 17, 18, 19; 20, 21, 22, 23; &c.; with the *general* expression corresponding to each column, $\left(\frac{X}{4}\right)_r = 0$, &c. We *know* that the Sunday Letters of 1600 were B A; which accordingly we write in the top line of the Sunday Letters, under 16, and in the horizontal row marked 00 (which express the two last figures of each centurial year). We then find in the regular order of succession all the Sunday Letters of that century, down to 1699; and which are shown in the first of the four columns. Having completed this column, we go to the second, answering to 1700, 2100, and generally to $\left(\frac{X}{4}\right)_r = 1$. These centurial years are all Common years, and therefore their Letter is C, which regularly follows D of 1699, and so on of the rest.

As the series of Sunday Letters recur on the same days of the month every 400 years, we may for the present purpose include *all* centuries under the symbols

$$\left(\frac{X}{4}\right)_r = 0, \left(\frac{X}{4}\right)_r = 1, \left(\frac{X}{4}\right)_r = 2, \left(\frac{X}{4}\right)_r = 3.$$

Hence the construction of the Table is obvious.

Units and Tens, by which the Given Year exceeds the Hundreds.				*Hundreds* of Years A. D. 16 20 24 $\left(\frac{X}{4}\right)_r = 0$	17 21 25 $\left(\frac{X}{4}\right)_r = 1$	18 22 26 $\left(\frac{X}{4}\right)_r = 2$	15 19 23 27 $\left(\frac{X}{4}\right)_r = 3$
00				BA	C	E	G
1	29	57	85	G	B	D	F
2	30	58	86	F	A	C	E
3	31	59	87	E	G	B	D
4	32	60	88	DC	FE	AG	CB
5	33	61	89	B	D	F	A
6	34	62	90	A	C	E	G
7	35	63	91	G	B	D	F
8	36	64	92	FE	AG	CB	ED
9	37	65	93	D	F	A	C
10	38	66	94	C	E	G	B
11	39	67	95	B	D	F	A
12	40	68	96	AG	CB	ED	GF
13	41	69	97	F	A	C	E
14	42	70	98	E	G	B	D
15	43	71	99	D	F	A	C
16	44	72		CB	ED	GF	BA
17	45	73		A	C	E	G
18	46	74		G	B	D	F
19	47	75		F	A	C	E
20	48	76		ED	GF	BA	DC
21	49	77		C	E	G	B
22	50	78		B	D	F	A
23	51	79		A	C	E	G
24	52	80		GF	BA	DC	FE
25	53	81		E	G	B	D
26	54	82		D	F	A	C
27	55	83		C	E	G	B
28	56	84		BA	DC	FE	AG

To illustrate the use of this Table, let us take a few examples.

Ex. 1. Find the Sunday Letter of A. D. 1700.

Look for 17 at the top, and for 00 in the side column, and where the horizontal row intersects the vertical column, we find C, the required Letter.

Ex. 2. Find the Sunday Letter of 1875.

The vertical column in which 18 is written intersects the horizontal row which contains 75, in C, which is the required Letter.

Ex. 3. Find the Sunday Letter of A. D. 8750.

Here $$\left(\frac{87}{4}\right)_r = 3:$$

we look, therefore, at the intersection of column 3 with the horizontal row containing 50, and we find A, the required Letter.

Ex. 4. Find the *Gregorian* Sunday Letter of 1582, after the correction.

Here $$\left(\frac{15}{4}\right)_r = 3:$$

therefore, where column 4 intersects horizontal row 82, we find C, the required Letter.

We know that B was the Sunday Letter of A. D. 1. Hence column 2 shows all the *Julian* Sunday Letters of the first century of our era; and, by a simple calculation, all the Julian Sunday Letters up to the date of the correction of the Calendar may be derived from that column. The Rule is this:—Divide the number of the given year by 28, and the Letter in column 2 standing opposite to the remainder will be the required Letter. If there be no remainder, the Letters will be DC.

Ex. 1. Required the (Julian) Sunday Letter for A. D. 1050.

Here we have $$\left(\frac{1050}{28}\right)_r = 14,$$

the horizontal line of which intersects column 2 in G, which is the required Letter.

Ex. 2. Required the Julian Sunday Letter for A. D. 1148.

Here $$\left(\frac{1148}{28}\right)_r = 0, \text{ or } 28:$$

therefore, the sought Letters are DC.

Hence it appears that this Table will give the Sunday Letters for both the Old and New Style.

96. To find for what centuries the Old and New Style Sunday Letters are the same, we remember (Art. 94) that they agree whenever

$$\odot = 4 + \text{mult. } (7). \qquad (1)$$

But we have seen (Art. 87, Note 3) that, when $\odot$ is given, we find σ by the formula

$$\sigma = \left(\frac{4\,(\odot + 12)}{3}\right)_w. \qquad (2)$$

Hence L and L' agree when

$$\sigma = \left(\frac{4\,(16 + \text{mult. } (7))}{3}\right)_w.$$

The successive values this last bracket takes are 21, 30, 39, 49, 58, &c. Hence L and L' coincide during the centuries whose centurial figures are 21, 30, 39, 49, 58, &c.

The *same* series of Letters do not repeat themselves until after a cycle of 2800 years, as we might conclude from the fact that 2800 is the L. C. M. of 700 (the Julian Cycle, Art. 30) and 400 (the Gregorian Cycle, Art. 95).

97. We can also construct another useful Table, to find the Gregorian Sunday Letter for any year A. D. x, from the general expression for L, Art. 93, (3): viz.,

$$L = \left(\frac{x + \left(\frac{x}{4}\right)_w - \sigma + \left(\frac{\sigma}{4}\right)_w}{7}\right)_r,$$

the scale being the Prayer Book scale.

The construction of the Table depends on the two following Lemmas, which are easily proved:—

(*a*). The term $\sigma - \left(\frac{\sigma}{4}\right)_w$, depending on the Solar Equation, is the same for any two centuries separated by an interval of 9, *provided we start from a suitable value of* σ.

(*b*). For any century whose centurial figures are divisible exactly by 4, the second term is the same as for the century immediately preceding.

Proof.—(*a*). Let $\sigma' = \sigma + 9$: then

$$\sigma' - \left(\frac{\sigma'}{4}\right)_w = \sigma + 9 - \left(\frac{\sigma + 9}{4}\right)_w$$

$$= \sigma + 9 - \left(\frac{\sigma}{4}\right)_w - \left(\frac{9}{4}\right)_w,$$

by Art. 87, Note 2, equation (c'), provided that σ is not $= 4m + 3$, in which case we should have

$$\left(\frac{\sigma + 9}{4}\right)_w = \left(\frac{\sigma}{4}\right)_w + \left(\frac{9}{4}\right)_w + 1.$$

But $$\left(\frac{9}{4}\right)_w = 2;$$

hence, omitting the 7, we find that, unless $\sigma = 4m + 3$,

$$\sigma' - \left(\frac{\sigma'}{4}\right)_w = \sigma - \left(\frac{\sigma}{4}\right)_w.$$

(*b*). Let $\sigma = 4m$: then $\sigma - 1 = 4m - 1$.

Thus

$$4m - \left(\frac{4m}{4}\right)_w = 3m; \text{ and } 4m - 1 - \left(\frac{4m-1}{4}\right)_w = 4m - 1 - (m - 1) = 3m.$$

This being premised, we have only to find the different values of L for *nine* centuries, beginning with 1600, and thence we ascertain the values for *all* centuries by means of Lemmas (*a*) and (*b*).

B	C	D	E	F	G	A
6	5	4	3	2	1	0
				1582 1600	1700	1800
1900 2000	2100	2200	2300 2400	2500	2600	2700 2800
2900	3000	3100 3200	3300	3400	3500 3600	3700
3800	3900 4000	4100	4200	4300 4400	4500	4600
4700 4800	4900	5000	5100 5200	5300	5400	5500 5600
5700	5800	5900 6000	6100	6200	6300 6400	6500
6600	6700 6800	6900	7000	7100 7200	7300	7400
7500 7600	7700	7800	7900 8000	8100	8200	8300 8400
8500	8600	8700 8800	8900	9000	9100 9200	9300
9400	9500 9600	9700	9800	9900 10000	&c.	

Explanation of the Table.—The numbers 0, ... 6 in the second row express the seven different values of second term of the formula above given, namely $\sigma - \left(\frac{\sigma}{4}\right)_w$. We see by Art. 93, that these values are

For 1600 . . . 2
1700 . . . 1
1800 . . . 0

For	1900, 2000	6
	2100	5
	2200	4
	2300, 2400	3
	2500	2
	2600	1
	&c.	

The vertical columns show all the centuries to which each of the seven values at the head of the columns respectively belong. It will be observed that the centurial figures in each vertical column differ by 9 (Lemma (a)), except when the centurial figure is exactly divisible by 4 (Lemma (b)), in which case they have the same head figure as the preceding century, *e. g.*

$$\frac{1900}{2000};\ \frac{4700}{4800};\ \&c.$$

This Table is the *First* of the three General Tables in the Prayer Book, entitled "A *General* Table for finding the Dominical or Sunday Letter." The Rule given in the Prayer Book is the translation into words of the formula (Art. 93 (3)), from which this Table is constructed, viz.: "To find the Dominical Letter for any given year of our Lord, add to the year its fourth part, omitting fractions $\left(x + \left(\frac{x}{4}\right)_w\right)$, and also the number which standeth at the top of the column, wherein the number of hundreds contained in that given year is found $\left(-\sigma + \left(\frac{\sigma}{4}\right)_w\right)$; divide the sum by 7, and if there be no remainder, then A is the Sunday Letter; but if any number remaineth, then the Letter which standeth under [*over* in the above Table] that number at the top of the Table is the Sunday Letter."

Ex. 1. Find Sunday Letters for 1680 (Leap-year).

Here we have

$$x + \left(\frac{x}{4}\right)_w = 2100$$

$$-\left(\sigma - \left(\frac{\sigma}{4}\right)_w\right) = -12 = +2,$$

$$L = \left(\frac{2102}{7}\right)_r = 2 = \text{F};$$

hence the Letters are GF.

Ex. 2. Find Sunday Letter for 1875.

Here we have $x + \left(\frac{x}{4}\right)_w = 2343$: and $-\left(\sigma - \left(\frac{\sigma}{4}\right)_w\right) = -14 = 0$.

Hence we get $L = \left(\frac{2343}{7}\right)_r = 5 = \text{C}$.

The same result may be obtained directly from (3) or (5), Art. 93.

98. To change Old Style into New, and *v. v.*, the following considerations must be attended to.

The number of days omitted in 1582 was *ten*. The Solar Equation drops three out of every four centurial Bissextile days, beginning with February 29, 1700. Accordingly, the N. S. date is $(10 + \odot)$ days in advance of O. S. date. Hence the general formula for converting O. S. into N. S. is this,

$$\text{N. S.} = \text{O. S.} + (10 + \odot) \text{ days.}$$

In other words, if d = New Style − Old Style,

$$d = 10 + \odot = 10 + \sigma - 16 - \left(\frac{\sigma - 16}{4}\right)_w.$$

By Art. 87, Note 3, we can solve for σ in terms of d, and get

$$\sigma = \left(\frac{4(d+2)}{3}\right)_w.$$

E. g., required the century in which the difference of style will be a whole year (365 days). Here $d = 365$, and $\therefore \sigma = 489$: *i. e.*, the Julian year 48,900 will begin on the same day as the Gregorian year 48,901.

The Solar Equation $(\odot)$, or the omitting of a Bissextile day, does not begin till 1700, when the Bissextile day is omitted, and March 1 follows February 28. A second Bissextile day is omitted in 1800; a third in 1900; and in 2000 there is no omission. Hence we have this

Rule for changing Old Style into New.

At and from	Oct. 5, 1582,	up to	Feb. 29, 1700 (inclusive),	add	10	days	to the O. S. date.
„	Mar. 1, 1700,	„	1800	„	11	„	„
„	„ 1800,	„	1900	„	12	„	„
„	„ 1900,	„	2100	„	13	„	„

and so on.

Thus, March 25, 1585,	O. S.,	is (adding 10) March 35 = April 4, N. S.	
Dec. 25, 1692,	„	„	Dec. 35 = Jan. 4, 1693, N. S.
Feb. 29, 1700,	„	is (adding 10, and omitting Biss.) = March 11, N. S.	
April 16, 1738,	„	is (adding 11) = April 27, N. S.	
Aug. 30, 1850,	„	is (adding 12) = Sept 11, „	

and so on.

The only difficulty that can present itself is when the given date is very near the point at which the Bissextile day occurs in the Old Style, but is omitted in the New.

The following Table will clear up all doubts in this case.

1700. O. S.		N. S.	1800. O. S.		N. S.
Feb. 28	=	March 10	Feb. 28	=	March 11
„ 29	=	„ 11	„ 29	=	„ 12
March 1	=	„ 12	March 1	=	„ 13

1900. O. S.		N. S.	2100. O. S.		N. S.
Feb. 28	=	March 12	Feb. 28	=	March 13
„ 29	=	„ 13	„ 29	=	„ 14
March 1	=	„ 14	March 1	=	„ 15

and so on.

Hence we see that the following is the

Rule for changing New Style into Old.

At and from	Oct. 15, 1582,	to Mar. 10, 1700 (inclus.),	subtract	10	days from the date.
„	Mar. 11, 1700	„ 11, 1800	„	11	„
„	„ 12, 1800	„ 12, 1900	„	12	„
„	„ 13, 1900	„ 13, 2100	„	13	„

Ex. 1. Find the O. S. date corresponding to March 9, 1700. Here, subtracting 10, we get the required date O. S., Feb. 27.

Ex. 2. Find the O. S. date corresponding to March 14, 2100. Here, subtracting 14, we get March 0, or Feb. 29, O. S.

99. We have already seen (Art. 36 (*e*)), how to find the number of days elapsed from 1st of Jan., A.D. 1, to 1st of Jan., A. D. x, x being any year OLD STYLE. We have now to inquire what modification in this calculation is necessary for *New Style*.

The formula referred to is $D = (x-1)\,365 + \left(\frac{x-1}{4}\right)_w$.

To adapt this to any year x, N. S., we must deduct the 10 nominal days dropped in 1582, and also the number of *Leap*-years omitted by the Solar Equation, viz.,

$$\sigma - 16 - \left(\frac{\sigma-16}{4}\right)_w.$$

So that, if Δ denote the number of days from A. D. 1 to any year x, N. S., we have, putting

$$\Delta = (x-1)\,365 + \left(\frac{x-1}{4}\right)_w - 10 - (\sigma - 16) + \left(\frac{\sigma-16}{4}\right)_w$$

$$= (x-1)\,365 + \left(\frac{x-1}{4}\right)_w (-\sigma - 2) + \left(\frac{\sigma}{4}\right)_w. \qquad (1)$$

Hence, $$\Delta = D + 2 - \left(\frac{x}{100}\right)_w + \left(\frac{x}{400}\right)_w. \qquad (2)$$

This is the formula for the number of days elapsed from the 1st of January, A. D. 1, to January 1 of any year subsequent to 1582.

Ex. For 1583. Here $\sigma - 2 = 13$, and $\left(\frac{\sigma}{4}\right)_w = 3$; and $-13 + 3 = -10$. Hence 10 must be subtracted from the number of Julian days.

To find number of days from Jan. 1, A. D. x, O. S., to Jan. 1, A. D. x', N. S., we use Art. 36, equation (2), and get

$$D' - D = (x' - x)\,365 + \left(\frac{x'-x}{4}\right)_w - (\sigma - 2) + \left(\frac{\sigma}{4}\right)_w \left[+1\right], \qquad (3)$$

where the quantity within the square brackets is to be added only when

$$\left(\frac{x'-1}{4}\right)_r < \left(\frac{x-1}{4}\right)_r.$$

Ex. Find the number of days from April 5, A. D. 30, to April 5, 1840.

Here $x' - x = 1810$. Hence, since April 5, A. D. 1, to April 5, 1840, equals from Jan. 1 to Jan. 1, *plus* the Leap-year in 1840, we get

$$D' - D = 661{,}103 \text{ days} - 12 = 661{,}091 \text{ days}.$$

In the same way, if we require the number of days from *any* given day, A. D. x (before correction), to any given day, A. D. x' (after correction), we get the answer from formula (3), Art. 36, by subtracting the number $\sigma - 2 - \left(\frac{\sigma}{4}\right)_w$.

100. The same modification must be made in calculating intervals by the method of quadriennia (Art. 39). After finding the number of days in the Julian reckoning, we must subtract the number $\sigma - 2 - \left(\frac{\sigma}{4}\right)_w$: that is to say, during the period from Oct. 5, 1582, to Feb. 29, 1700, we must subtract 10 days: and so on as explained in Art. 98.

Take, for example, the instance given above, viz.: the number of days from April 5, A. D. 30, to April 5, 1840.

Here we have $\frac{1810}{4} = 452 + 2$; of which two, one—1840—is Bissextile.

400 (Quadr.)	=	584,400	days.
50 „	=	73,050	„
2 „	=	2,922	„
1 × 365	=	365	„
1 × 366	=	366	„
		661,103	= total number of Julian days;

from which subtract $\sigma - 2 - \left(\frac{\sigma}{4}\right)_w = 12$, which gives the same number of days as before.

If we require the number of days elapsed between any two dates, *both* of which are subsequent to the correction in 1582:—then if x_1, x_2 be the years, and σ, σ' the corresponding centuries and a', a, the days of the month, formula (3), Art. 36, becomes

$$D' - D = (x_2 - x_1) \times 365 + \left(\frac{x_2 - 1}{4}\right)_w - \left(\frac{x_1 - 1}{4}\right)_w - (\sigma' - \sigma) + \left(\frac{\sigma'}{4}\right)_w - \left(\frac{\sigma}{4}\right)_w + a' - a. \quad (4)$$

101. The Gregorian Calendar, or New Style, was immediately adopted in Spain and Portugal, and the greater part of Italy; in France, two months after the publication of the Bull. The Catholic Cantons of Switzerland adopted it in 1583, the Poles in 1586, and Hungary in 1587. In Germany, the Emperor and the Catholic states adopted it in 1583; but the Elector of Saxony, and the other Protestant states, refused to do so. The reason why the Protestants refused at first to adopt the Gregorian reformation was not merely their objection to yield to the mandate of the Pope, but also because Joseph Scaliger and other learned authorities had objected to the new reckoning as not sufficiently in accordance with astronomical accuracy (1). The difference of reckoning thus arising between the adherents of New and Old Style gave rise to many disputes and perplexities, especially in places where the Protestants and Roman Catholics were mixed together. At Augsburg, in the Diet, for many years disputes on the subject took place, which were known as the "*Calendar controversy*" (Kalenderstreit) (2). At last, in the year 1700, at the instance of Leibnitz, and

with the aid of the mathematician Weigel, the German Protestant states reformed their Calendar by omitting 11 days, and so far agreeing with the Gregorian reformation. The Festival of Easter, however, was determined, not by the fourteenth day of the *Calendar* Moon, but by the true astronomical Full Moon of the Tables. This reformed German Calendar was adopted by Denmark, the United Netherlands, and the Reformed Cantons of Switzerland. The different modes of determining Easter necessarily occasioned, from time to time, a difference in the day kept, and consequently, new disputes. The first case of this kind happened in 1724, when the astronomical reckoning of the Full Moon gave the 9th of April as Easter Day, while the Calendar Moon gave the 16th. A second difference took place in 1744, when the German Protestants kept the 29th of March, and the Roman Catholics April 5. A third would have occurred in 1778, and a fourth in 1798, had not the *Corpus Evangelicorum*, at the instance of Frederick the Great, decided to restore the old method of calculating Easter by the Calendar Moon. In 1752, England adopted the Gregorian Calendar ([3]). And now the Russians and Greeks are the only nations in Europe who adhere to the Old Style. At present their reckoning is 12 days behind that of the rest of Europe. Throughout Eastern Christendom also the Old Style still prevails.

(1). It was to meet such objections that Clavius wrote his great work, so often referred to. Still, after all his labour and skill in defence of the Gregorian reformation, Roman Catholic authorities themselves admit its defects.

(2). See on the subject of the Kalenderstreit, *Ideler*, ii. 321–325; *Piper*, Geschichte des Osterfests Zeit der Kalenderreformation, Berlin, 1845; and *Herzog*, Encyclopädie, Art. "Kalender."

(3). In the twelfth century, the Anglican Church commenced to date the year from the 25th of March. This practice was adopted by civilians in the fourteenth century, and continued in use until the reformation of the Calendar in 1752.

In fact, prior to the Statute of 24 Geo. II. ch. 23, there were two different commencements of the year in England:—

1°. The *Historical* year began on January 1.

2°. The *Ecclesiastical*, *Civil*, and *Legal* year began on the 25th of March.

Great confusion arose, as might be expected, from these different modes of computation. The Legislature, the Church, and civilians, referred every event occurring between January 1 and March 25 to a different year from the historians. Sir H. Nicolas (*Chronology of History*, p. 42) observes that two of the most celebrated events in English history afford remarkable examples of the confusion arising from this difference of reckoning. Most authorities state that Charles I. was beheaded on the 30th of January, 1648; while others, with equal correctness, assign that event to the 30th of January, 1649. Again, the revolution which drove James II. from the throne is stated by some writers to have taken place in February, 1688; whilst, according to others, it happened in February, 1689. These discrepancies are explained by the fact that some historians used the *Civil* and *Legal* year, while others used the *Historical*. Any event occurring after the 25th of March, in either of those years, would have been referred to the same year by all. To avoid the mistakes which this difference of reckoning produced, it was usual, in giving the date of any event between the 1st of January and the 25th of March, to write *both* years, the *Legal* and the *Historical*, the latter being put *after* the former, thus, January 30, 1648–9.

At the introduction of the New Style into England, and for some time after, it was usual to express the two dates in documents and books, by writing them in the form of a fraction, the old above, the new below the line, thus: $\frac{19}{30}$ May, 1753; $\frac{\text{25th Dec., 1753}}{\text{6th Jan., 1754}}$. During the nineteenth century, there are *twelve* days' difference between the Old and New Styles; so that Christmas Day, Old Style, corresponds to the 6th of January, the Festival of the Epiphany, in the New. Hence, the latter day is often called "Old Christmas Day."

A trace of the old legal commencement of the year still remains in the custom of dating leases from the 25th of March (Lady Day), and making it one of the gale days for payment of rents.

It is worthy of remark, that, while the Ecclesiastical year was reckoned to commence on the 25th of March, during the whole interval from the first compiling of our Prayer Book, in 1549, down to its final revision, in 1662, and while in the Prayer Books of 1604 and 1662 the year is *expressly* stated to begin on March 25 (*vid.* Keeling, Liturg. Britann., p. xi. n.), still the Lessons in the daily Calendar are arranged with reference to January 1.

102. I have just said that the Gregorian reformation was not adopted in England until the middle of the eighteenth century. An effort to do so was made in the reign of Queen Elizabeth, about two years after the issue of the Bull in 1582. A Bill passed two readings in the House of Lords, entitled "An Act to give Her Majesty authority to alter and new make a Calendar, according to the Calendar used in other countries." No further notice seems to have been taken of it. Popular prejudice appears to have been strongly against it, chiefly, it would seem, because the plan had emanated from Rome. A mathematician named H. Wilson wrote an able pamphlet in 1735 very conclusively proving the necessity of the reform. It was not, however, until the year 1751 that an Act of Parliament (24 Geo. II. ch. 23) was passed, entitled "An Act for regulating the commencement of the year (1), and for correcting the Calendar now in use." The preamble recites that the legal supputation of the year of Our Lord in England, according to which the year began on March 25 (2), had been found to be attended with many inconveniences, and that the Julian Calendar had been discovered to be erroneous; so that the Vernal Equinox, which, at the time of the Council of Nice, happened on or about the 21st of March, then happened on the 9th or 10th of that month; and that the error was still increasing; and that a method for the correcting of the Calendar had been received and established, and was then generally adopted by almost all the other nations of Europe. It was therefore enacted—

1°. That the supputation according to which the year of Our Lord began on the 25th of March shall not be used after the last day of December, 1751; and that the 1st day of January next following shall be reckoned as the first day of the year 1752, and so in all future years (3).

2°. That the natural day next following the 2nd of September, 1752, shall be called and reckoned as the 14th of September, omitting the 11 intermediate nominal days of the Common Calendar (4). Thus, September 3–13, 1752, had no existence in the English Calendar, as October 5–14, 1582, had none in the Roman.

3°. That the centurial years 1800, 1900, 2100, 2200, 2300, &c., shall be Common years; and that 2000, 2400, 2800, &c., shall continue to be reckoned Bissextile.

4°. That the method hitherto received in England for the calculation of the Paschal Full Moons, having become considerably erroneous, shall be discontinued; and that from and after September 2, 1752, Easter Day and the Moveable Feasts depending upon it shall be celebrated according to the new Tables and Rules annexed to the Act; and that a new Calendar shall be drawn, to be prefixed, along with the Tables and Rules, to the Book of Common Prayer.

These Rules and Tables were adopted and confirmed in Ireland by Statute 21 and 22 Geo. III. ch. 48, in the year 1782, exactly 200 years after the Gregorian correction. So that of all the countries in Europe which have adopted the Gregorian reformation, Ireland is the last.

(1). This Act may be seen in Mr. A. J. Stephens' edition of the Book of Common Prayer, vol. i., p. 274; and Sir H. Nicolas' "Chronology of History," p. 37.

(2). The Anglo-Saxons, according to Bede, began their year with Christmas Day, December 25. But in the twelfth century, the Anglican Church began to date the year from March 25; which practice was adopted by civilians in the fourteenth century, and continued in use until the Statute of 24 Geo. II., in 1751.

In fact three different commencements of the year have been in use in England:—

1°. The *Historical year*, which has for a very long time begun on *January* 1;

2°. The *Civil, Ecclesiastical, and Legal year*, which began on *December* 25, and continued to be used to the end of the thirteenth century; after which, in the fourteenth century, the year commenced on March 25, and so continued till 1752.

(3). The result of this clause in the Act was that the year 1751 (the English *Annus confusionis*) contained only 282 Ecclesiastical and Civil days: viz., 365 *minus* the 83 from January 1 to March 24, inclusive, which, according to the old reckoning, belonged to the year 1750. This was, therefore, the shortest year in the Ecclesiastical and Civil annals of England.

The reason why 11 days were omitted is obvious. The Gregorian correction in 1582 omitted 10 days; and during the 170 years that elapsed between that and 1752, the Solar error of the Julian Calendar had accumulated to another day. In the Gregorian Calendar, this additional error was prevented by the year 1700 being reckoned as a Common year, instead of a Bissextile.

(4). We have seen (Art. 76) that, in the Gregorian correction, *October* (4–15) was selected for the omission of the 10 days, because of the few Saints' days occurring in it. Some such reason probably determined the omission of the 11 days in *September* (3–13), no holiday intervening. Holy Cross Day (14th)

was probably retained because it was a festival much observed both in the East and West; and in England there were no less than 106 churches under the designation either of Holy Rood or Saint Cross. It is also the day in reference to which the September Ember Days are determined.

103. The leading part in carrying the famous Act of Parliament was taken by the celebrated Lord Chesterfield. The London mob were so enraged by the omission of the 10 days, by which they thought their lives were shortened, that they pursued his carriage through the streets, loudly clamouring for the restitution of the days of which he had dared to rob them. The death of the Astronomer Royal (Dr. Bradley), who had prepared the new Tables for the Government, and which took place shortly after the passing of the Act, was commonly regarded as a Divine judgment upon him for his iniquity in shortening the lives of so many people [1]. It is remarkable that, in the Tables and Rules drawn up in 1752, to adapt the English Ecclesiastical reckoning to the Gregorian reckoning, no use whatever is made of the *Epacts*, the substitution of which for the Golden Numbers of the Old Calendar formed one of the characteristic features of the Gregorian reformation. Bradley, who drew up the Rules and Tables, adhered to the old method of calculation by the Golden Numbers; and the Third *General* Table in the Prayer Book was intended, by means of the Golden Numbers, to correspond to the Extended Table of Epacts in the Gregorian system, of which more hereafter (Art. 123).

(1). In some parts of England it was believed that, at the moment when Christmas Day began, the cattle always went down on their knees in their stalls; and it is said that, when the change of style was introduced, the cattle refused to acknowledge it, but kept the commencement of the old Christmas Day in the old fashion. Nor were these objections peculiar to Protestant countries. In Roman Catholic countries, however, the authority of the Pope removed all obstacles. The learned Jesuit, Riccioli, gravely informs us that the blood of St. Januarius, which liquefied on the 19th of September, and a supernatural rod which always budded on the morning of Christmas Day, both acknowledged the change of style as soon as ever it was made.—De Morgan, *l. c.*, p. 19, Note.

104. The effect of omitting 11 nominal days in 1752 was, as in the case of the Gregorian correction in 1582, to change the Sunday Letter for all Sundays subsequent to the 14th of September. That year was Leap-year, and the Letters were E D. Eleven Calendar Letters were passed over, which changed the second Letter from D to A ($11 = 7 + 4$) [1]. A, therefore, became the Sunday Letter for the remainder of the year. Hence, the year 1752 had, in the English Calendar, *three* Sunday Letters, and in this respect it is unique, viz.:—

E, from Jan. 1 to Feb. 29;
D, „ March 1 „ Sept. 2;
A, „ Sept. 14 „ end of the year.

In English history, the *Old Style* is used for any year *before* 1752; and *in* 1752, up to September 2, inclusive: but from September 14 (inclusive) the *New Style* is used.

(1). The Julian Sunday Letters for 1752 being E D, and the Calendar Letter of September 2 being G, that day was Wednesday. The next day, Thursday, was reckoned as the 14th, the Calendar Letter of which is E: consequently, the following Sunday had A for its Letter.

The formula for the Gregorian Sunday Letter in the eighteenth century—viz.,

$$L = \left(\frac{n + \left(\frac{n}{4}\right)_w + 1}{7}\right)_r \text{ (Art. 92)}$$

(Prayer Book scale)—holds, of course, for the English Calendar after the correction in 1752. Thus, for instance, this formula gives for 1752, *after* the correction, $L = 0 = A$.

105. Let us now specially consider the change made by the Gregorian reformation in the Old Paschal Table (Art. 60); in other words, let us see what form the *Paschal Table* assumes for the year 1583. As the correction did not take place till October, 1582, the Easter of that year was past, and had been kept in accordance with the old Table. Now, we have seen (Art. 80) that the result of *the clearing off the accumulations* of Solar and Lunar errors was to *depress the Golden Numbers* 7 days from the positions which they occupied in the Old Church Calendar. In order, therefore, to find the *Paschal Table* for the year 1583, we have only to shift, in the Table, Art. 60, all the 19 Golden Numbers of the Paschal Lunation (29 days) 7 places downwards. The Golden Numbers denote, as in the old Table, the Paschal *Full* Moons. In this way we get

THE NEW PASCHAL TABLE,

AFTER THE GREGORIAN CORRECTION OF THE CALENDAR UNTIL A. D. 1699.

Golden Numbers denoting Paschal Full Moons.	Days on which Easter can fall.	Sunday Letters.	Golden Numbers denoting Paschal Full Moons.	Days on which Easter can fall.	Sunday Letters.
III.		C	XV.	April 8	G
	March 22	D	IV.	,, 9	A
XI.	,, 23	E		,, 10	B
	,, 24	F	XII.	,, 11	C
XIX.	,, 25	G	I.	,, 12	D
VIII.	,, 26	A		,, 13	E
	,, 27	B	IX.	,, 14	F
XVI.	,, 28	C		,, 15	G
V.	,, 29	D	XVII.	,, 16	A
	,, 30	E	VI.	,, 17	B
XIII.	,, 31	F	XIV.	,, 18	C
II.	April 1	G		,, 19	D
	,, 2	A		,, 20	E
X.	,, 3	B		,, 21	F
	,, 4	C		,, 22	G
XVIII.	,, 5	D		,, 23	A
VII.	,, 6	E		,, 24	B
	,, 7	F		,, 25	C

As there was no *Solar* Equation until A. D. 1700, and the Lunar Equation did not begin till 1800, this Table was in use from 1583 to 1699, both inclusive. In other words, the Golden Numbers continued in the same places during that period, no change being required.

The Rule for finding the *Sunday Letters* varied from the Rule given in the old Table, in consequence of the omission of the 10 days in 1582. The Rule for the interval now under consideration (1583–1699) is the following (Art. 91):—To the number of the year add its fourth part, omitting fractions, and to the result add 2. Divide the sum by 7; if there be no remainder, the Sunday Letter is A; but if there be a number remaining, the Letter corresponding to that number in the annexed small Table will be the required one.

A	0
G	1
F	2
E	3
D	4
C	5
B	6

In Leap-years, the Letter found as above will be the *second* Letter, the first being the preceding one in the scale.

Ex. 1. Find the Sunday Letter for A. D. 1605.

The Rule gives 6 as the remainder, which, by the scale, corresponds to B.

Ex. 2. Find the Sunday Letters for A. D. 1608 (Leap-year).

The Rule gives the remainder 3 = E (the second Letter). Therefore, the required Letters are F E.

To find Easter Day by this Paschal Table for the year 1645. The Golden Number

$$= \left(\frac{1645 + 1}{19}\right)_r = \text{XII.}$$

The Sunday Letter, by the Rule, = 3 = A.

Now, in the Table, Golden Number XII. is opposite to April 11, and the next A is opposite to April 16; which, therefore, is the day required.

This Table can, of course, be exhibited in the same form as the Old Paschal Table (Art. 61). This corrected form of the old Table is that in our Prayer Book, entitled "A Table to find Easter till the year 1899, inclusive."

106. In 1800, the Solar and Lunar Equations both took place, and neutralized each other; so that the two Tables hold good until A. D. 1899. In 1900, the Solar Equation alone will take place, the effect of which will be to depress all the Golden Numbers one place lower down in the month. The Table, so modified, will hold good until 2199; because there is neither Solar nor Lunar Equation in 2000, and, therefore, no change takes place in that century.

This Paschal Table is also found in our Prayer Book, entitled "TABLE TO FIND EASTER DAY FROM THE YEAR 1900 TO THE YEAR 2199, INCLUSIVE."

The formula for the Sunday Letter from 1800 to 1899, inclusive, is

$$L = \left(\frac{x + \left(\frac{x}{4}\right)_w}{7}\right)_r = \bar{x} + 0 \quad \text{(Art. 97)};$$

and from 1900 to 2099,

$$L = \left(\frac{x + \left(\frac{x}{4}\right)_w + 6}{7}\right)_r = \bar{x} + 6 \quad ,, \quad ;$$

and, finally, from 2100 to 2199,

$$L = \left(\frac{x + \left(\frac{x}{4}\right)_w + 5}{7}\right)_r = \bar{x} + 5 \quad ,, \quad .$$

From what has just been said, it is obvious that we can construct all the successive *Paschal Tables* which correspond with the changes required by the Solar and Lunar Equations. It is only necessary, for this purpose, to keep in mind the general Rule (Art. 82): "When the *Solar* Equation alone takes place, the Golden Numbers *descend* one place; when the *Lunar* Equation alone takes place, they *ascend* one place; when *neither* takes place, or *both* concur, there is no displacement."

In this way, I have drawn up the *thirty different* Paschal Tables required from the correction of the Calendar (1583) down to A. D. 8499, both inclusive—a period of 6917 years. The *thirty-first* Table, beginning with A. D. 8500—or 6900 years, reckoning from 1600—is the same, with regard to the position of the Golden Numbers, as the first (from 1583–1699); all the Golden Numbers having during the interval successively occupied all the 29 days of the Paschal Lunar Month. Thus, Golden Number III., which, in the first reformed Paschal Table, in 1583, stands at the top of the column, opposite March 21, is found at the bottom of the thirtieth column, opposite April 18; and in the thirty-first column (or Paschal Table) III. is again at the top, having thus returned to its old position. And so the Golden Numbers will proceed through another cycle, ending A. D. 15,399; and so on, as already explained in Art. 86. There are, therefore, only thirty such Tables. The line of figures, from 0 to 30, at the top and bottom of the rows of vertical columns, denote, respectively, the number of displacements of the Golden Numbers downwards from their position in 1600; and are, in fact, taken from the *Equation Table* (Art. 83). They may be called the *Index Figures* to the columns of the Golden Numbers, as they correspond to the *Index Letters* of the Expanded Table of Epacts. And the dates at the top of the Table show, respectively, the centuries for which each column of Golden Numbers holds good. For example, *after* the correction in 1582, and *before* any *secular* displacement arose—in other words, when the *Index* Figure is 0—we find under 0 the corresponding Paschal Table, already given in Art. 105; and above it we see that it held good in Roman Catholic countries, from our 1582 to 1699. Similarly, when one displacement downwards took place, we see under Index Figure 2 the corresponding Paschal Table; and at the top we learn that it holds from 1700 to 1899. It is the first of the temporary Paschal Tables given in the Prayer Book, and did not come into use in the English Church until September, 1752. Index Figure 2 again denotes the column of Golden Numbers that will be in use from 1900 to 2199—the second temporary Paschal Table given in the Prayer Book: and so on of all the rest. These Index Figures, as already remarked, are the same as the Index Figures in the *second* General Table in the Prayer Book.

Suppose, again, we desire to know which of the thirty Paschal Tables will be in use in any century, we look for that century in the top of the Table, and the corresponding column gives the required Table. *E. g.*, required the Paschal Table for the century beginning A. D. 6000. We find that the same Paschal Table will hold from 5900 to 6199, so that the column whose index is 19 will be the required Table. If the given century for which the corresponding Paschal Table is required be beyond this Table, we have only to seek, by formula (2), Art. 89, what is the total displacement for that century (σ), and divide it by 30; then look out the Index Figure in the Table which is equal to the remainder, and the corresponding column will be the Paschal Table required. *E. g.*, required the Paschal Table for A. D. 11,600. Here we have the remainder 13; hence, the required Table is that under Index Figure 13.

The first column in this Table shows the *Old Paschal Table* before the reformation of the Calendar; so that this Table exhibits all the Paschal Tables from the first settlement of one in the Ancient Church down to the latest century in which the Gregorian reckoning shall continue in use.

The following Table, above referred to, gives a SYNOPTICAL VIEW OF THE THIRTY DIFFERENT PASCHAL TABLES required by the Gregorian reckoning.

Paschal New Moons.	Paschal Full Moons.	Cal. Letters.	Old Style.	1582–1699	1700–1899	1900–2199	2200–2299	2300–2399	2400–2499	2500–2599	2600–2899	2900–3099	3100–3399	3400–3499	3500–3599	3600–3699	3700–3799	3800–4099	4100–4199	4200–4499	4500–4699	4700–4999	5000–5099	5100–5199	5200–5299	5300–5399	5400–5699	5700–5899	5900–6199	6200–6299	6300–6399	6400–6499	6500–6599	6600–6699	6700–6799	6800–6899	6900–6999	7000–7299	7300–7499	7500–7799	7800–7899	7900–7999	8000–8099	8100–8199	8200–8499	8500–8699
				0	1	2	3	4	3	4	5	6	7	8	9	8	9	10	11	12	13	14	15	16	15	16	17	18	19	20	21	20	21	22	23	22	23	24	25	26	27	28	27	28	29	30
March 8	March 21	C	16	3	14		6	17	6	17		9		1	12	1	12		4	15		7	18		18		10		2	13		13		5	16	5	16		8			11		11		3
,, 9	,, 22	D	5		3	14		6		6	17		9		1		1	12		4	15		7	18	7	18		10		2	13	2	13		5		5	16		8	19		19		11	
,, 10	,, 23	E		11		3	14		14		6	17		9		9		1	12		4	15		7		7	18		10		2		2	13		13		5	16		8	19	8	19		11
,, 11	,, 24	F	13		11		3	14	3	14		6	17		9		9		1	12		4	15		15		7	18		10		10		2	13	2	13		5	16		8		8	19	
,, 12	,, 25	G	2	19		11		3		3	14		6	17		17		9		1	12		4	15	4	15		7	18		10		10		2		2	13		5	16		16		8	19
,, 13	,, 26	A		8	19		11		11		3	14		6	17	6	17		9		1	12		4		4	15		7	18		18		10		10		2	13		5	16	5	16		8
,, 14	,, 27	B	10		8	19		11		11		3	14		6		6	17		9		1	12		12		4	15		7	18	7	18		10		10		2	13		5		5	16	
,, 15	,, 28	C		16		8	19		19		11		3	14		14		6	17		9		1	12	1	12		4	15		7		7	18		18		10		2	13		13		5	16
,, 16	,, 29	D	18	5	16		8	19	8	19		11		3	14	3	14		6	17		9		1		1	12		4	15		15		7	18	7	18		10		2	13	2	13		5
,, 17	,, 30	E	7		5	16		8		8	19		11		3		3	14		6	17		9		9		1	12		4	15	4	15		7		7	18		10		2		2	13	
,, 18	,, 31	F		13		5	16		16		8	19		11		11		3	14		6	17		9		9		1	12		4		4	15		15		7	18		10		10		2	13
,, 19	April 1	G	15	2	13		5	16	5	16		8	19		11		11		3	14		6	17		17		9		1	12		12		4	15	4	15		7	18		10		10		2
,, 20	,, 2	A	4		2	13		5		5	16		8	19		19		11		3	14		6	17	6	17		9		1	12	1	12		4		4	15		7	18		18		10	
,, 21	,, 3	B		10		2	13		13		5	16		8	19	8	19		11		3	14		6		6	17		9		1		1	12		12		4	15		7	18	7	18		10
,, 22	,, 4	C	12		10		2	13	2	13		5	16		8		8	19		11		3	14		14		6	17		9		9		1	12	1	12		4	15		7		7	18	
,, 23	,, 5	D	1	18		10		2		2	13		5	16		16		8	19		11		3	14	3	14		6	17		9		9		1		1	12		4	15		15		7	18
,, 24	,, 6	E		7	18		10		10		2	13		5	16	5	16		8	19		11		3		3	14		6	17		17		9		9		1	12		4	15	4	15		7
,, 25	,, 7	F	9		7	18		10		10		2	13		5		5	16		8	19		11		11		3	14		6	17	6	17		9		9		1	12		4		4	15	
,, 26	,, 8	G		15		7	18		18		10		2	13		13		5	16		8	19		11		11		3	14		6		6	17		17		9		1	12		12		4	15
,, 27	,, 9	A	17	4	15		7	18	7	18		10		2	13	2	13		5	16		8	19		19		11		3	14		14		6	17	6	17		9		1	12	1	12		4
,, 28	,, 10	B	6		4	15		7		7	18		10		2		2	13		5	16		8	19	8	19		11		3	14	3	14		6		6	17		9		1		1	12	
,, 29	,, 11	C		12		4	15		15		7	18		10		10		2	13		5	16		8		8	19		11		3		3	14		14		6	17		9		9		1	12
,, 30	,, 12	D	14	1	12		4	15	4	15		7	18		10		10		2	13		5	16		16		8	19		11		11		3	14	3	14		6	17		9		9		1
,, 31	,, 13	E	3		1	12		4		4	15		7	18		18		10		2	13		5	16	5	16		8	19		11		11		3		3	14		6	17		17		9	
April 1	,, 14	F		9		1	12		12		4	15		7	18	7	18		10		2	13		5		5	16		8	19		19		11		11		3	14		6	17	6	17		9
,, 2	,, 15	G	11		9		1	12	1	12		4	15		7		7	18		10		2	13		13		5	16		8	19	8	19		11		11		3	14		6		6	17	
,, 3	,, 16	A		17		9		1		1	12		4	15		15		7	18		10		2	13	2	13		5	16		8		8	19		19		11		3	14		14		6	17
,, 4	,, 17	B	19	6	17	17	9		9		1	12	12	4	15	4	15	15	7	18	18	10		2		2	13	13	5	16	16	16	16	8	19	8	19	19	11		3	14	3	14	14	6
,, 5	,, 18	C	8	14	6	6	17	9	17	9	9	1	1	12	4	12	4	4	15	7	7	18	10	10	10	10	2	2	13	5	5	5	5	16	8	16	8	8	19	11	11	3	11	3	3	14
Line of Equations, or Number of Descents,*				0	1	2	3	4	3	4	5	6	7	8	9	8	9	10	11	12	13	14	15	16	15	16	17	18	19	20	21	20	21	22	23	22	23	24	25	26	27	28	27	28	29	30

* N. B.—This and the top line of figures may be called *Index* Figures to the columns of Golden Numbers above them, as they correspond to the *Index Letters* in the Expanded Table of Epacts.

It is important to bear in mind that the movement downwards of the Golden Numbers is one common to all, each being simultaneously advanced, their *relative* intervals as regards each other being undisturbed. It is also necessary to remember that the motion is a circular one; each Golden Number, after reaching the lowest point, April 18, going up to the top again. We may illustrate the movement by that of an endless chain of 29 links, moving uniformly from left to right; one link being dropped from a pin at the top at the beginning of each period denoted by the corresponding Index Figure.

We see that the Paschal Table corresponding to Index 23 (6900–6999) is the same as the *Old* perpetual Paschal Table.

These thirty almanacs occupy *forty-two* columns, because of twelve Index Figures being each repeated twice, by reason of the Lunar Equation occurring alone, and so causing a *regression.*

107. It will also be observed that the last two rows, corresponding to April 17 and 18, are much more crowded than the rest. The reason of this is that, in the Old Church Calendar, the Paschal Lunation consisted of only 29 days. In other words, there were but 29 days contained within the two Paschal Full Moon limits—March 21, and Apri 18; and within those 29 days all the 19 Golden Numbers, indicating the Paschal Full Moons, were necessarily contained. The same rule respecting the Paschal Lunation was for the most part continued in the Gregorian Calendar. In the latter there are but two Paschal Lunations with 30 days. In the Old Paschal Table (column 1, Table, Art. 106), beginning with Golden Number XVI., all the Golden Numbers were contained exactly within the Table, the last (VIII.) falling on April 18. But in the correction of 1582—when all the Golden Numbers were simultaneously moved down seven places—III., which, in the old Table, was opposite April 13, was transferred to March 21; and VI., which, in the old Table, was at April 10, was transferred to April 17. XIV. was still wanting in column 0, and, its place in the old Table being April 12, when moved down *seven* places, it would come to April 19; but as this was outside the extreme Paschal limit, it was necessary to move it up one place—viz., to April 18. Hence, there are three consecutive Golden Numbers at the bottom of column 0. Similarly, in 1700, when the Solar Equation required a descent of one place, XIV. came up to March 21, and XVII., VI., moved down to April 17, 18. But in 1900, when another descent will be necessary, XIV. will move down to March 22; IX. will move from April 15 to April 16; XVII. *cannot* move down, because VI. also must have a place: so that XVII. and VI. cannot move in column 2; and there are again three consecutive Golden Numbers at the bottom of this column. The result of all this is that the row correspond-

ing to April 18 becomes entirely filled with Golden Numbers; and the row opposite April 17 becomes crowded with numbers, contiguous to each other, that cannot move down, because April 18 is full.

It will further be observed that these *duplicate* Golden Numbers in the penultimate row (April 17) are the following, taken in the order of occurrence:—XVII., XII., XV., XVIII., XIII., XVI., XIX., XIV., that is to say, the *last eight* of the Golden Numbers (XII.-XIX.) We shall have occasion to revert to this, when we come to speak of the Epacts, and the Extended Table of Epacts.

108. The thirty Paschal Tables which have just been considered are, in effect, nothing more than the development of the *third of the General Tables* in our Prayer Book, taken in connexion with the *second* General Table (or Equation Table). It may be called the *Expanded* Table of Golden Numbers. This Table III. was drawn up by Dr. Bradley, as a substitute for Lilio's *Extended Table* of Epacts (*vid.* Art. 123). In fact, Table III. exhibits in terms of the Golden Numbers what the "Extended Table" does in terms of the Epacts.

TABLE III.

Paschal Full Moon.	Calendar Letter.	The Golden Numbers.																		
		I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.	XIX.
Mar. 21	C	8	19	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26
,, 22	D	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27
,, 23	E	10	21	2	13	24	5	16	27	8	19	0	11	22	3	14	25	6	17	28
,, 24	F	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29
,, 25	G	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	0
Mar. 26	A	13	24	5	16	27	8	19	0	11	22	3	14	25	6	17	28	9	20	1
,, 27	B	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2
,, 28	C	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	0	11	22	3
,, 29	D	16	27	8	19	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4
,, 30	E	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5
Mar. 31	F	18	29	10	21	2	13	24	5	16	27	8	19	0	11	22	3	14	25	6
Apr. 1	G	19	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7
,, 2	A	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8
,, 3	B	21	2	13	24	5	16	27	8	19	0	11	22	3	14	25	6	17	28	9
,, 4	C	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10
Apr. 5	D	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	0	11
,, 6	E	24	5	16	27	8	19	0	11	22	3	14	25	6	17	28	9	20	1	12
,, 7	F	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13
,, 8	G	26	7	18	29	10	21	2	13	24	5	16	27	8	19	0	11	22	3	14
,, 9	A	27	8	19	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15
Apr. 10	B	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16
,, 11	C	29	10	21	2	13	24	5	16	27	8	19	0	11	22	3	14	25	6	17
,, 12	D	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18
,, 13	E	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19
,, 14	F	2	13	24	5	16	27	8	19	0	11	22	3	14	25	6	17	28	9	20
Apr. 15	G	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21
,, 16	A	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	0	11	22
,, 17	B	5	16	27	8	19	0	11	22	3	14	25	6	17	28	9	20	1	12	23
,, 17	B												7	18	29	10	21	2	13	24
,, 18	C	6	17	28	9	20	1	12	23	4	15	26								
Apr. 18	C	7	18	29	10	21	2	13	24	5	16	27	8	19	0	11	22	3	14	25

109. The construction and use of this Table are easily understood.

The first column contains all the days on which the Paschal Full Moon can fall, from March 21 (the earliest) to April 18 (the latest), both inclusive. The second column contains the *Calendar* Letters corresponding to these days, respectively. Then follow 19 other columns, each column having one of the Golden Numbers at its head, and in regular order, from I. to XIX. Each of these 19 columns contains 30 numbers, from 0 to 29:

so that there are altogether *thirty horizontal rows* of these figures. But as there are only 29 days within the Paschal limits (both included), the twenty-eighth row is interposed between the twenty-seventh and twenty-ninth, and so divided that the first 11 numbers of it belong to April 18, and the last eight to April 17. By this means, the 29 descents which each Golden Number makes in a *full* Lunar month are provided for. We shall recur to this presently, after we have explained the way in which these 30 rows of figures are constructed.

It is obvious, on inspection, that all these rows are consecutively evolved from the first, by the continued addition of a unit. Hence, we have only to ascertain how the first row is constructed.

In the Paschal Table, after the correction in 1582, and which Table served, as we have seen, until 1699, the Golden Number III., denoting the Paschal *New* Moon, was found at March 8; and, denoting the Paschal *Full* Moon, at March 21. Accordingly, as there was *no* change in 1600, 0 was placed under Golden Number III., in the first row, in the same horizontal line as March 21, to show that from the year 1600 to 1699, Index 0 (Table II., Prayer Book), any year whose Golden Number was III. would have its Paschal Full Moon on March 21. In the next year (IV.) of that Lunar Cycle, the Calendar Moon was *eleven* days past its full on March 21; and, therefore, 11 is written immediately after 0, and under IV. In the following year (V.) of that cycle, the Calendar Moon was 22 days past full on March 21, and, accordingly, 22 is written after 11 and under V. And so on to the end of that cycle of 19 years, 11 being continuously added to each preceding number in the row, and 30 being dropped when the sum exceeds 30 (Art. 115). The only exception is in the case of passing from Golden Number XIX., at the end of one cycle, to I. at the beginning of the next cycle, when 12 must be added instead of 11 (*vid.* Art. 114). Having thus formed the topmost row of horizontal figures, the second row is formed from it by simply adding a unit to each figure in the former. And, following the same process, all the rows, down to the last, are successively evolved. The figures in any row show the number of descents made respectively by the Golden Numbers standing at the head of each column, and what is the date of the corresponding Paschal Full Moon. For example, in the fifth row, we find 4, under Golden Number III., indicating that Golden Number III. has descended 4 places below its position in 1600, and that the corresponding Paschal Full Moon for any year during which Index 2 holds good, and whose Golden Number is III., will fall on March 25.

The inspection of the Table shows that, in any two adjacent columns, the same figures are separated by an interval of 19 places, reckoned downwards in the vertical, except in the case of the two columns corresponding to XIX. and I., where the interval is 18, instead of 19. In counting these 19 places, we must remember that the bottom of one column is supposed to be continuous with the top of the next. For example, 0, in column IV., is 19 places lower than 0 in column III.: and there is the same interval between any two consecutive Golden Numbers in the same Paschal Calendar (*vid.* Art. 53, 8°). Hence, it appears that, for the year 1600, and the entire interval between 1583 and 1699, during which the Index 0 is in use, the places of the 19 Golden Numbers in the Paschal Table 0 are given by the days of the month, respectively, on the same horizontal lines as the 19 cyphers 0. For example (see Table, Art. 106), 0, in the column belonging to Golden Number III., is on the same horizontal row with March 21; and, therefore, III. is to be affixed to March 21 in the Paschal Table whose Index is 0. In the column belonging to IV., 0 is found in the same horizontal row with April 9: therefore, IV. is to be affixed to April 9, in the Paschal Table whose Index is 0; and so on, until all the 19 Golden Numbers are assigned to their proper days in the Paschal Table. The Paschal Table thus constructed by means of Table III., and the Index Number 0, will be found to be identical with that in Art. 106, Index 0. Similarly, in order to construct the Paschal Table belonging to Index 1, or after one descent, we have only to look out in Table III. for the figure 1, in each of the 19 columns of Golden Numbers, and the corresponding days of the month will be the places of the Golden Numbers, respectively. Thus, III. will belong to March 22, IV. to April 10; and so on.

The Table so constructed is that under Index Figure 1, in Art. 106. Hence, we see the reason of the Rule affixed to the General Table II. in the Prayer Book (the *Equation* Table of the Golden Numbers).

TABLE II.

I.	II.	III.	I.	II.	III.
	Years of our Lord.			Years of our Lord.	
B	1600	0	B	5200	15
	1700	1		5300	16
	1800	1		5400	17
	1900	2		5500	17
B	2000	2	B	5600	17
	2100	2		5700	18
	2200	3		5800	18
	2300	4		5900	19
B	2400	3	B	6000	19
	2500	4		6100	19
	2600	5		6200	20
	2700	5		6300	21
B	2800	5	B	6400	20
	2900	6		6500	21
	3000	6		6600	22
	3100	7		6700	23
B	3200	7	B	6800	22
	3300	7		6900	23
	3400	8		7000	24
	3500	9		7100	24
B	3600	8	B	7200	24
	3700	9		7300	25
	3800	10		7400	25
	3900	10		7500	26
B	4000	10	B	7600	26
	4100	11		7700	26
	4200	12		7800	27
	4300	12		7900	28
B	4400	12	B	8000	27
	4500	13		8100	28
	4600	13		8200	29
	4700	14		8300	29
B	4800	14	B	8400	29
	4900	14		8500	0
	5000	15		&c.	
	5100	16			

To find the month and days of the month to which the Golden Numbers ought to be prefixed in the Calendar, in any given year of our Lord, consisting of entire hundred years, and in all the intermediate years betwixt that and the next hundredth year following, look in the second column of Table II. for the given year, consisting of entire hundreds, and note the number or cypher which stands against it in the third column; then, in Table III. look for the same number in the column under any given Golden Number, which, when you have found, guide your eye sideways to the left hand, and in the first column you will find the month and day to which that Golden Number ought to be prefixed in the Calendar, during that period of one hundred years.

The Letter B, prefixed to certain hundredth years in Table II., denotes those years which are still to be accounted Bissextile or Leap-years in the New Calendar; whereas all the other hundredth years are to be accounted only Common years.

As the Index Figure includes 1700 and 1800, we see that the Paschal Table constructed for Index 1 extends from 1700 to 1899.

It is, of course, easy to construct in the same way the Paschal Table corresponding to Index Figure 2: and as this figure embraces three centuries, the Paschal Table will extend from 1900 to 2199.

If the year A. D. be given, and we desire to know the corresponding Paschal Table, we have only to observe the Rule given above from the Prayer Book.

As Table III. contains 30 different rows, and therefore 29 descents for each Golden Number, not including the position 0, which each Golden Number held, A. D. 1600, it follows that this Table, in connexion with Table II., will give 30 different Paschal Tables, as we have already found in Art. 106.

110. Reference has already been made to the twenty-eighth row of figures in Table III., which are broken up into two sets, the first eleven being referred to April 18, and the last eight to April 17. But it may be well to say a few words more in explanation of this matter.

When, in the course of the secular descents, any one of the Golden Numbers (which properly indicate the *New* Moons) comes to April 5th, the corresponding *Full* Moon (14th day after New) falls on April 19th; which being *beyond the limit* (18th), the Golden Number must be put back to the 18th. Now, in the *distribution* of the Golden Numbers in the Calendar, the first eight Golden Numbers are *found immediately* after the last eight, thus :—

XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.	XIX.
I.	II.	III.	IV.	V.	VI.	VII.	VIII.

Hence, when any of the first eight reaches April 5, the corresponding Full Moon falls on the 19th, and therefore that Golden Number must be moved back to the 18th. But this place is already occupied by the Number *immediately* above it. Now, it was *a fixed rule* in the Old Calendar, which was retained in the New also, that two Golden Numbers should never fall on the *same day of the month* within the *same cycle*. Consequently, the upper of the two contiguous Numbers must also be put back one place, viz., to April 17, in order to make room for its comrade Number. To take, for example, the first case which presents itself in the expanded Table (Art. 106), viz., the column whose Index Figure is 2. Here, Golden Number VI. (*Full* Moon) fell on April 19, and had to be moved back to April 18. But as XVII. immediately precedes VI., in the Calendar, its place should be the 18th. Consequently, to avoid a collision with VI., it was moved back to the 17th, and similarly of the seven other Golden Numbers from XII. to XIX. It is always possible to move back one of these to the 17th of April, without intruding on a place already occupied by another Golden Number, because there is always a vacant space between any *contiguous pair* and the Golden Number which precedes them: *e.g.* $\frac{\text{XVII.}}{\text{VI.}}$ is separated from IX. by a vacant place.

This completely explains the broken line in Table III. We shall find that a similar explanation holds good in the case of the extended Table of Epacts (Art. 123).

111. This Table III., though framed for a perpetual *Paschal* Table only, may yet be extended (like the extended Table of Epacts) to the *whole year*, for any number of centuries, by means of the auxiliary Table II.; because, the places of the *Paschal* Golden Numbers (New or Full Moons) for any century being given, they may be continued *backwards* to the beginning of the year, and *forwards* to the end. For example, in the *Paschal* Table for Golden Number III., 0 (1582–1699), derived from Table III., the Golden Number affixed to March 21 is III. Therefore, by the law of sequence of the Golden Numbers, XIV. is to be affixed to March 20, VI. to March 18, XVII. to March 17, and so on *backwards* to the beginning of January. Similarly, in the same Paschal Table, XIV. is affixed to April 18 (the other limit of the Paschal Lunation); after which III. must be affixed to April 19, XI. to April 21, and so on to the *end* of December.

As in the Old Calendar the Golden Numbers denoted New, not Full, Calendar Moons, I have prefixed to this Table the date of the *New* Moon corresponding to each Full Moon; which will facilitate the extension of the New *Paschal* Tables to the *whole* year: especially when we bear in mind that the several Golden Numbers in January fall on the same days of the month as in March. For example, if we desire to construct the *whole* Calendar for the year 1700, we see by the *Paschal* Table that Golden Number XIV. denoting *Full* Moon falls on March 21, and the corresponding *New* Moon on March 8, and therefore on January 8. Having thus the place of one Golden Number in January, we can assign the places of all the Golden Numbers for the whole year.

112. I have hitherto dealt with the Gregorian Calendar merely as a continuation of the Old Church Calendar: the accumulated errors arising from the inaccurate length of the Julian year, and of the Lunar Cycle being corrected, and rules laid down for the periodical adjustment of them in future. I have adhered to the old terminology, and especially to the mode of determining the Calendar New (and Full) Moons by the Golden Numbers. When the Gregorian reformation was adopted in England (1752) the old system and its terms were retained, and were still continued in our Prayer Book. But the Gregorian reformation made an important change in respect of the use of the Golden Numbers, which I now proceed to explain.

We have seen that the accumulated errors of the Julian year required the moving *down* of the Golden Numbers 10 days from the places which they occupied in the Old Church Calendar; and that the accumulated error of the Lunar Cycle required that they should be placed 3 days higher up; thus giving, as the result of both, a *descent* of 7 places. We have also seen that at each occurrence of the *Solar* Equation, the Golden Numbers must be moved *down* one place; while, on the other hand, at each occurrence of the Lunar Equation, they must be moved *up* one place. So that the Golden Numbers would require to be shifted in the Calendar almost every century. To avoid this inconvenience, and the possible mistakes which in different centuries might arise from this constant shifting of the Golden Numbers, Lilius (who, as we have seen, constructed the Gregorian Calendar) rejected the Golden Numbers altogether, and substituted in their place another set of numbers with a fixed relation to them, and which therefore answer the same purpose of indicating the Calendar New and Full Moons, while they do not shift their places in the Calendar. These new numbers are the EPACTS. Reference has been made to them already (Art. 63); but for our present purpose it is necessary to enter more fully into the subject.

113. The word *Epacts* (ἐπακταί, *sc.* ἡμέραι) is of Greek origin (ἐπάγειν, *addere, intercalare*), and was used in general to denote the excess of the greater of two given periods of time over the lesser. It was, however, specially and most usually applied to designate the *eleven days* excess of the common Solar year of 365 days over the common Lunar year of 354 days (Ideler, ii., p. 239, N.) ([1]). It was, in this way, used to express the age of the Calendar Moon on some given day of the Solar year, especially the 1st of January or the 31st of December. In the Latin *Fasti Consulares* ([2]), the work of an unknown author in the fourth century, extending over a period of 862 years, viz., from U. C. 246 to U. C. 1107, or A. D. 354, Epacts are used to denote the *Moon's age on January 1st.* The 84-year cycle is employed, and the first year of the cycle the Epact is I. That is to say, a New Moon was supposed to fall on January 1, at the commencement of the 862 years of this Table (*vid.* Ideler, ii., 239–40).

On the other hand, in the Paschal Tables of Cyril, Bishop of Alexandria, who flourished in the fifth century, and also in those of Dionysius Exiguus (sixth century), and Bede (eighth century), the term *Epacts* was employed to denote the *Moon's age on the 22nd of March* (Ideler, ii., 293), the earliest day on which Easter can fall ([3]). These authors adopted the Alexandrian calculation, according to which a New Moon fell on the 23rd of March, A. D. 323, and which was taken as the *first year* or Epoch of the 19-year Lunar Cycle. Accordingly, on March 22, the Moon's age, or Epact, was 30 days, or 0 ([4]). The following year (the second of the cycle) the March New Moon fell 11 days earlier, viz., on the 12th; so, the Epact was 11. The year after (third of the cycle) the New Moon fell again 11 days earlier, viz., on March 1; and the Epact was 22; and so on throughout the cycle. Consequently, the *relation between the Golden Numbers and the Dionysian Epacts* (as before defined) is as follows,

G. N.	I.,	II.,	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.	I.	&c.
Epact.	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18.	0.	

It is obvious that the law of formation of these Epacts is given by the formula

$$\left(\frac{11\,(N-1)}{30}\right)_r$$

where N is the Golden Number of the year.

We have already seen (Art. 56) that as a cycle began A. D. 323, another cycle began the year before the Christian era, in other words, the year of the Nativity. Accordingly, this is the epoch from which in the Church Calendar the calculation of the Golden Numbers begins, so that the Rule to find the Golden Number for any year A. D. x is

$$\left(\frac{x+1}{19}\right)_r.$$

(1). It would seem that ἐπακταί (sc. ἡμέραι), first denoted the 11 days *added* to the Lunar year of 354 days in order to complete the Solar year of 365. Isidore, Orig., lib. vi., c. 17, says:—"Epactas Græci vocant, Latini *adjectiones* annuas Lunares, quæ per *undenarium* numerum usque ad tricenarium in se resolvuntur."

(2). The *Fasti Consulares* were first brought to light by Cardinal Noris, from a MS. in the Royal Library at Vienna.

(3). Bede (*De Tempor. Rat.*, c. 48) says, "Quæ in circulo decemnovali adnotatæ sunt Epactæ Lunam quota sit in xi. Cal. April., ubi Paschalis est Festi principium, signant" (*Ideler*, ii. 261).

(4). When the Calendar Moon has *fully* completed its period of 30 days, its age may be considered as 0, as it has *then* ceased to exist and a *New* Moon begins.

114. The authors of the Gregorian Calendar *revived* the old definition of the Epact, viz., the Calendar Moon's age at the beginning of any Solar year, or at the end of the preceding year. And Lilius devised a very ingenious mode of substituting in the Calendar the Epacts for the Golden Numbers, in order to indicate the Calendar New Moons, and, at the same time, without a constant shifting of them from the Secular Equation. To explain this fully, it will be well to commence by setting forth the Calendar as Lilius constructed it, and which is known as

The Perpetual Gregorian Calendar, or, the Epact Calendar.

(Clavius, p. 40; Delambre, Astron. Mod., i. p. 42; Encyclop. Britt., Art. "Calendar.")

Days.	Jan.		Feb.		Mar.		Apr.		May.		June.		July.		Aug.		Sept.		Oct.		Nov.		Dec.		Days.
	ε.	λ.	ε.	λ.	ε.	λ.	ε.	λ.	ε.	λ.	ε.	λ.	ε.	λ.	ε.	λ.	ε.	λ.	ε.	λ.	ε.	λ.	ε.	λ.	
1	•	A	29	D	•	D	29	G	28	B	27	E	26	G	25 24	C	23	F	22	A	21	D	20	F	1
2	29	B	28	E	29	E	28	A	27	C	25' 26	F	25' 25	A	23	D	22	G	21	B	20	E	19	G	2
3	28	C	27	F	28	F	27	B	26	D	25 24	G	24	B	22	E	21	A	20	C	19	F	18	A	3
4	27	D	25' 26	G	27	G	25' 26	C	25' 25	E	23	A	23	C	21	F	20	B	19	D	18	G	17	B	4
5	26	E	25 24	A	26	A	25 24	D	24	F	22	B	22	D	20	G	19	C	18	E	17	A	16	C	5
6	25' 25	F	23	B	25' 25	B	23	E	23	G	21	C	21	E	19	A	18	D	17	F	16	B	15	D	6
7	24	G	22	C	24	C	22	F	22	A	20	D	20	F	18	B	17	E	16	G	15	C	14	E	7
8	23	A	21	D	23	D	21	G	21	B	19	E	19	G	17	C	16	F	15	A	14	D	13	F	8
9	22	B	20	E	22	E	20	A	20	C	18	F	18	A	16	D	15	G	14	B	13	E	12	G	9
10	21	C	19	F	21	F	19	B	19	D	17	G	17	B	15	E	14	A	13	C	12	F	11	A	10
11	20	D	18	G	20	G	18	C	18	E	16	A	16	C	14	F	13	B	12	D	11	G	10	B	11
12	19	E	17	A	19	A	17	D	17	F	15	B	15	D	13	G	12	C	11	E	10	A	9	C	12
13	18	F	16	B	18	B	16	E	16	G	14	C	14	E	12	A	11	D	10	F	9	B	8	D	13
14	17	G	15	C	17	C	15	F	15	A	13	D	13	F	11	B	10	E	9	G	8	C	7	E	14
15	16	A	14	D	16	D	14	G	14	B	12	E	12	G	10	C	9	F	8	A	7	D	6	F	15
16	15	B	13	E	15	E	13	A	13	C	11	F	11	A	9	D	8	G	7	B	6	E	5	G	16
17	14	C	12	F	14	F	12	B	12	D	10	G	10	B	8	E	7	A	6	C	5	F	4	A	17
18	13	D	11	G	13	G	11	C	11	E	9	A	9	C	7	F	6	B	5	D	4	G	3	B	18
19	12	E	10	A	12	A	10	D	10	F	8	B	8	D	6	G	5	C	4	E	3	A	2	C	19
20	11	F	9	B	11	B	9	E	9	G	7	C	7	E	5	A	4	D	3	F	2	B	1	D	20
21	10	G	8	C	10	C	8	F	8	A	6	D	6	F	4	B	3	E	2	G	1	C	•	E	21
22	9	A	7	D	9	D	7	G	7	B	5	E	5	G	3	C	2	F	1	A	•	D	29	F	22
23	8	B	6	E	8	E	6	A	6	C	4	F	4	A	2	D	1	G	•	B	29	E	28	G	23
24	7	C	5	F	7	F	5	B	5	D	3	G	3	B	1	E	•	A	29	C	28	F	27	A	24
25	6	D	4	G	6	G	4	C	4	E	2	A	2	C	•	F	29	B	28	D	27	G	26	B	25
26	5	E	3	A	5	A	3	D	3	F	1	B	1	D	29	G	28	C	27	E	25' 26	A	25' 25	C	26
27	4	F	2	B	4	B	2	E	2	G	•	C	•	E	28	A	27	D	26	F	25 24	B	24	D	27
28	3	G	1	C	3	C	1	F	1	A	29	D	29	F	27	B	25' 26	E	25' 25	G	23	C	23	E	28
29	2	A			2	D	•	G	•	B	28	E	28	G	26	C	25 24	F	24	A	22	D	22	F	29
30	1	B			1	E	29	A	29	C	27	F	27	A	25' 25	D	23	G	23	B	21	E	21	G	30
31	•	C			•	F			28	D			25' 26	B	24	E			22	C			19' 20	A	31

The structure of this Calendar is very simple, and, indeed, obvious on mere inspection. The first column contains the days of the month; the following columns the twelve months of the *Common year* of 365 days. The column of figures in each month are the Epacts (ε), and the corresponding Letters are the Calendar Letters (λ). There are *thirty* (1) Epacts, numbered from 1 to 30, or 0, which Clavius writes * (2); and, with the exceptions to be presently mentioned, one of the Epacts is written opposite each of the 365 days of the year, beginning with January 1. They are written in groups of thirty, each group beginning with 30 or *, and ending with 1; a new group commencing where the preceding one ended. Thus, the first group ends with January 30, and the next begins with January 31; and so on down to December 31, to which the Epact 20 is affixed. It was desired to make *each group of thirty include a Lunar month;* therefore, as the Lunar Calendar months are alternately 30 and 29 days in length, it was necessary in the hollow months to reduce the thirty Epacts to twenty-nine, which was done by assigning two consecutive Epacts to the same day of the month. The Epacts which Clavius preferred to double, for this purpose, were 24 and 25. Accordingly, as the above Calendar shows, the Epacts 24 and 25 are *both* written opposite February 5, April 5, June 3, August 1, September 29, November 27; these being the six hollow months. Lilius, the author of the new Calendar, had proposed (*vid.* Clavius, p. 5) a different mode of this "Equation of the Lunar months;" viz., writing the *two* Epacts ω (the same as Clavius' *) and 29 opposite the 31st of January, 31st of March, 29th May, 27th June, 24th September, 22nd November (3). The reason why Clavius substituted for this pair of double Epacts the pair 25, 24, was, that by this means the Lunations according to the cycle of 30 Epacts, and especially the *Paschal* Lunations, were rendered more conformable to the Lunations of the Golden Numbers. For, as appears from the inspection of the Old and New Calendars, the Equation of the Lunar Month is made by the Golden Numbers in the former in almost exactly the same places as by the Epacts 25 and 24 in the latter (4).

If the Equation of the Lunar month were made (as Lilius proposed) at the end of March, there would be 7 Paschal Lunations of 30 days each, contrary to the usage of the Ancient Church; whereas, by making it (as Clavius does) on April 5, only two such Lunations can occur, viz., when both the Epacts 25 and 24 are in use. There will then be thirty, because in the case of 25, the New Moon is made to commence a day sooner (viz., 4th April, Epact 26); and in case of 24, because 24 is placed opposite April 5, instead of 6 (*vid.* Clavius, cap. x., § 10).

(1). As the Moon's age on any 1st of January exceeds by 11 days its age on the preceding January 1, the limit of the Moon's age being 30 days, it follows that the Epacts are formed from Epact 0, by the successive addition of 11, dropping 30 as fast as it arises. Now, an arithmetical series whose first term is 0, and common difference 11 (30 being dropped as often as it arises)—in other words, whose law of formation is $\left(\frac{n \times 11}{30}\right)_r$ (where n is any number from 0 to 29)—will give 30 distinct terms, and no more. This series, written out at length, is as follows:—

0	11	22	3	14	25	6	17	28	9	20
1	12	23	4	15	26	7	18	29	10	21
2	13	24	5	16	27	8	19	0		

(2). The reason why Clavius writes * in place of 30 and 0 is the following. The Epact being defined to be the number of days which remain over and above after the Lunation ending in December, it may happen that *two* such Lunations may occur—one ending on December 1, and the next on December 31. In the former case, 30 days will remain over, and the Epact will, accordingly, be 30; in the latter, no day remains, and the Epact will be 0. In order, then, that there may be some *one* Epact to express *both* these cases, a conventional sign (*) was employed; that is to say, * stands equally for 30 and 0. Delambre denies the necessity of any distinction between 30 and 0, and always writes 0 where Clavius employs * (*vid.* Clavius, cap. ix. § 14).

(3). It is obvious that, so far as the mere reduction of the 30 Epacts to 29, in the hollow months, is concerned, *any* pair of adjacent Epacts might be taken. Clavius chose 24 and 25, because he wished to adhere as closely as possible to the places of the Golden Numbers in the Old Calendar. Lilius had fixed on 29 and 30, thus making the reduction at the *beginning* of each hollow Lunar month; in other words, on January 31, March 31, May 29: an arrangement which would cause the Lunations in the New Calendar to deviate from those of the Old (*vid.* Clavius, cap. x. §§ 8, 9, 10).

(4). There is a remarkable diversity in the case of October. In the Old Calendar, the crowding of the Golden Numbers takes place at the beginning of the month, while in the New the Equation takes place at the end.

115. It is easy to see, by inspection of the Table, how the Epacts thus arranged indicate the Calendar New Moons throughout the successive years of any Lunar Cycle. Let us take, for example, a cycle in the third year of which a Calendar New Moon falls on January 1, as was the case in the Old or perpetual Julian Calendar (Art. 53) (1). We see that the asterisk (*) is affixed to January 1, January 31, March 1 and 31, April 29, and so on, down to December 21; and these are the very same days to which, in the Old Calendar, the Golden Number III., denoting the New Moons that year, is affixed. Hence we see that the Epact * indicates all the New Moons in the Gregorian Calendar, just as Golden Number III. did in the Old. The following year, the fourth of the Lunar Cycle, the Calendar Moon will be 11 days old on January 1; this Lunation, being a *full* one (because the last Lunation of the preceding year, November 22 to December 20, was *hollow*), will end on January 19 (11 + 19 = 30). Accordingly, the first New Moon of this fourth year falls on January 20, opposite which we find Epact 11;

and on the same day (January 20) we find Golden Number IV. affixed in the Old Calendar. And, in the same way, we find Epact 11 affixed to same days throughout that year as Golden Number IV. is in the Old Calendar—viz., February 18, March 20, April 18, May 18, &c. So that Epact 11, in the Gregorian Calendar, indicates the New Moons exactly as Golden Number IV. did in the Old. The fifth year of the cycle, the Moon is 22 days old on January 1, and the Lunation (being again a full one) ends January 8. The New Moon falls on January 9, opposite which we find Epact 22, and in the Old Calendar Golden Number V. Similarly, the other days to which Epact 22 is attached—viz., February 7, March 9, April 7, &c.—are those to which Golden Number V. is affixed in the Old Calendar; and, therefore, the New Moons are indicated by the Epact 22, just as by Golden Number V. Proceeding in the same way through the successive years of that cycle, we find that the Epact of each year denotes the Calendar New Moons of that year, just as the Golden Number. And the whole series of Epacts corresponding to the 19 Golden Numbers of that cycle, beginning with III., is as follows:—

G. N.	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.,	I.,	II.
	*	11,	22,	3,	14,	25,	6,	17,	28,	9,	20,	1,	12,	23,	4,	15,	26,	8,	19.

This series we have already found (Art. 63). The general formula of it is

$$\left(\frac{11\,(N-3)}{30}\right)_r = \left(\frac{11 \times N - 3}{30}\right)_r.$$

It will be observed that each Epact is formed from the next preceding by adding 11, 30 being dropped as fast as it arises, with a single exception—viz., in passing from Golden Number XIX. to Golden Number I. (or from one cycle to the next), 12 (not 11) is added to the Epact of XIX. The reason of this exception is that the Embolismic month added at the end of the cycle (Art. 50) contains only 29 days, instead of the usual 30. Therefore, after the addition of 11 to the Epact of Golden Number XIX., only 29 days (not 30) should be subtracted. But as the Computists preferred to keep up the uniformity of 30 days for all the 7 Embolismic months, they treated the 29 days as 30, and effected the compensation by adding 12 instead of 11 $(29 + 12 = 30 + 11)$(²).

(1). A New Moon (as has been just explained) fell on March 1, in the third year of the Lunar Cycle; and all the Calendar Moons in January fall on the same days as in March.

(2). We have already noticed (Art. 50) that when, as in the case under consideration, Golden Number III. coincides with January 1, the Embolismic month of 29 days belongs properly to the nineteenth and last year of the cycle beginning with Golden Number III., January 1; in other words, that it belongs to Golden Number II., not to Golden Number XIX. But the ancient Computists preferred that the number

12 should be added when the Golden Number was XIX., as being the greatest of the series; and they, accordingly, did not make the change (from 11 to 12) when the Golden Number was II. No notable error can arise in the Lunations from that transposition. Accordingly, Clavius, out of respect to antiquity, adopted the same course in all cases where the Golden Number is XIX., whatever Epact may correspond to it; even though that year is not always the last Embolismic year, nor even an Embolismic year at all (*Clavius*, cap. xii. § 13).

116. The series of Epacts just given in Art. 115 held good for all the years in which the Old Calendar was in use. But these Epacts were not employed for finding the Calendar New Moons, because the Golden Numbers from which the Epacts were derived were equally available for the purpose. But when the Gregorian correction was made in 1582, causing a displacement of the Calendar New Moons 7 days lower down, a new relation was established between the Golden Numbers and the Epacts. The Golden Number of that year was VI.—

$$\left(\frac{1582 + 1}{19}\right)_r = 6.$$

In the unreformed Calendar, VI. is found at October 20, but after the correction the corresponding New Moon was transferred to October 27. Now, in the Epact Calendar (Art. 114), we find that the Epact opposite to October 27 is 26, while that corresponding to October 20 (Golden Number VI.) is 3. That is to say, whereas the Epact corresponding to VI. before the correction was 3, the Epact representing the New Moon after the correction was 26. Similarly, the relations between the rest of the Golden Numbers and their corresponding Epacts before the correction, as shown in the series, Art. 115, were all changed; and the *new series of Epacts, after the correction in* 1582, became

G. N.,	III.	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.	XIX.,	I.,	II.
Epacts,	23,	4,	15,	26,	7,	18,	29,	10,	21,	2,	13,	24,	5,	16,	27,	8,	19,	1,	12.

(*D*)

Comparing this with the series of Art. 115, we see that the change in the Epacts consisted in the *subtraction of the same number*, 7, from all the pre-reformation Epacts. In fact, as the New Moons represented by the Old Style Golden Numbers all fell 7 days later after the correction, the Epacts corresponding to those corrected New Moons will be 7 places lower down, and therefore (as the Epacts are in *retrograde* order) 7 units less than before [1].

Thus, *without shifting the Golden Numbers from their old places, the New Moons* of the years corresponding to them, respectively, *were, after the correction, denoted by a new series of Numbers* having a given relation to the former.

(1). This relation between the Golden Numbers and the Epacts after the correction in 1582 may also be easily shown as follows. The December Calendar New Moon, indicated by the Golden Number VI. of that year, fell on December 18. But, by the lowering of the Golden Numbers 7 days, the New Moon fell on December 25. Consequently, the Moon was 7 days old on the 1st of January, 1583, the *seventh* year of the cycle. In other words, 7 was the Gregorian Epact of 1583. Starting, then, from this Epact, and Golden Number VII., 7, and forming the series of Epacts by the usual law of continued addition of 11, dropping 30 as it arises, we get, for 1583,

G. N.	VII.,	VIII.,	IX.,	X.,	. . . . .	XIX.,	I.,	II.,	III.,	IV.,	V.,	VI.
Epact,	7,	18,	29,	10,	. . . . .	19,	1,	12,	23,	4,	15,	26.

But no further change took place in 1583, so that this holds also for 1582, after the correction.

117. The above series (arrived at in Art. 116), which I shall call (D), for a reason to be presently noticed, *expressing the relation between the Golden Numbers and the Epacts after the correction in* 1582, is a very important one, inasmuch as it is, in fact, the *fundamental series of the Gregorian Calendar*, on which the relation between the Golden Numbers and Epacts for any year whatever, subsequent to the correction, depends. It is, therefore, desirable to obtain a general formula by which the above relation may be compendiously expressed.

This formula may be easily investigated as follows:—

Before the change, it was given by—

$$\epsilon = \left(\frac{11\,N - 3}{30}\right)_r,$$

where ϵ is Epact, N Golden Number.

The effect of the change was to lessen every Epact by 7; so from 1582 till 1699 the Epact is given by

$$\epsilon = \left(\frac{11\,N - 10}{30}\right)_r, \text{ or } \left(\frac{N + 10\,(N - 1)}{30}\right)_r. \qquad (a)$$

118. If there were no *secular* adjustments to be made after the correction in 1582, then series (D), with its formula (a), would hold good for ever. But, as we have seen, the effect of the Solar and Lunar Equations is to require a centurial change to be made in the places of the Golden Numbers; or, as in the Gregorian Calendar, these retain their places unchanged, in the Epacts; each descent of the Golden Number in the old reckoning requiring a *diminution* (on account of the *retrograde* arrangement) of a unit in the Epact. Since, as we have seen, there is no secular Equation in 1600, and, therefore, the places of the Calendar New Moons remain unchanged till 1699, there will be no change in the line of Epacts in (D), which will serve till 1699, inclusive. But in 1700 there was

a Solar Equation, and the Calendar New Moons all descended one day, and therefore all the Epacts also diminished by a unit. So that, for the year 1700, the Epacts in the fundamental series (D) must be lowered by a unit; and the series becomes :—

G. N.	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.,	I.,	II.	(C)
Epacts,	22,	3,	14,	25,	6,	17,	28,	9,	20,	1,	12,	23,	4,	15,	26,	7,	18,	*	11.	

This series of Epacts, which we will call (C), will continue in use as long as the corresponding Calendar Moons continue without change—that is, as we have seen (Art. 106), till 1899. In 1900, the Solar Equation again causes a descent of one place, and, therefore, a diminution of a unit in the above series of Epacts; and we get a new series :—

G. N.,	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.,	I.,	II.	(B)
Epacts,	21,	2,	13,	24,	5,	16,	27,	8,	19,	*	11,	22,	3,	14,	25,	6,	17,	29,	10.	

In the year 2000 there is neither Solar nor Lunar Equation; in 2100 there are both, and they neutralize each other; consequently, (B) holds up to 2199. In 2200 the Solar Equation takes place, and will require another diminution of the Epacts by a unit; and the series will become

G. N.,	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.,	I.,	II.	(A)
Epacts,	20,	1,	12,	23,	4,	15,	26,	7,	18,	29,	10,	21,	2,	13,	24,	5,	16,	28,	9.	

This series of Epacts holds up to 2299.

In 2300, the Solar Equation only takes place, requiring another diminution of the Epacts by 1, which gives

G. N.,	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.,	I.,	II.	(u)
Epacts,	19,	*,	11,	22,	3,	14,	25,	6,	17,	28,	9,	20,	1,	12,	23,	4,	15,	27,	8.	

This series of Epacts will hold up to 2399, inclusive.

In 2400 the Lunar Equation alone will take place, which *raises* the Calendar New Moons one place, and, therefore, the Epacts in (u) must all be *increased* by 1; which will bring back the series of Epacts to (A). In 2500 the Solar Equation alone occurs, and (u) will again come into use. By proceeding in this way, *thirty different series of Epacts will come successively into use*, corresponding to the thirty different places that the Calendar New Moons may occupy in the full Lunar Month of 30 days. These thirty different series of Epacts may all be exhibited in one Table, which is known as the *Extended Table of Epacts*, and which is strictly analogous to the *Extended Table of Golden Numbers*, Art. 106, and to the Prayer Book Table III.

119. But before exhibiting this Table, it will be desirable to draw up some points

connected with the *Perpetual Epact Calendar* (Art. 114), and which have not yet been noticed.

1°. It appears from inspection that all the 30 Epacts are written *twelve* times in the Calendar, with the exception of the eleven * . . . . 20, which are repeated a thirteenth time, viz., on the 11 last days of December, having occurred for the first time on the 11 opening days of January.

2°. It will be observed that in those six places in the Gregorian Calendar where the Epacts 25 and 24 are written together, the Epact 25′ (with a dash above it for distinction) is written on the same line with Epact 26. The reason of this is that the Epacts 25 and 24 both occur in the *same cycle* of 19 Epacts, when the Golden Number is any of the eight last, from XII.–XIX. (1) Consequently, in these cases, two Lunations, denoted respectively by these two Epacts, would fall on one and the same day, and the two corresponding Golden Numbers would similarly belong to the same day, contrary to the fundamental principle of the 19-year cycle, according to which the New Moons cannot return to the same days of the month until after the lapse of 19 years. This contradiction is escaped, by taking the Epact 26 instead of 25, in the cases just enumerated: in other words, whenever the Epact 25 belongs to a Golden Number greater than XI., the Epact 24 will also be in use in that cycle; therefore we must use 26, or the day opposite 26, as the day of the Calendar New Moon, *instead of* 25. By this means there will be no repetition during the cycle, of a New Moon on the same day, because 26 and 24 never occur together, when 25 and 24 do. To indicate this substitution of 26 for 25, the latter has a dash over it, and is written in the same line as 26. But whenever in any cycle the Epact 25 runs with any Golden Number less than XII. (*i.e.* from I. to XI.), it will never happen in that cycle that 24 is in use along with 25; accordingly, 25 is to be used, not 26.

(1) This appears clearly on the face of the Extended Table of Epacts (Art. 123).

120. It is easy to prove (what has been above stated) that,

First, in the first 11 years of any cycle, the Epacts 24 and 25 are not both in use together;

Secondly, in the last 8 years they are;

Thirdly, in this case 24 and 26 are not both in use.

Write the line of Epacts corresponding to the New Moon on January 1 in two groups as follows:—

*, 11, 22, 3, 14, 25, 6, 17, 28, 9, 20.
1, 12, 23, 4, 15, 26, 7, 18,

where the first group corresponds to the first eleven years of the cycle, and the second group to the last eight.

It appears, then, that in the first line there are no consecutive numbers, nor in the second; while all the numbers of the second line are, as far as they go, consecutive with those of the first. And the same thing will of course happen when both lines are increased by the addition of the same number. But by adding the numbers 1, 2, ... 19, successively, to both lines, and rejecting 30 whenever it can be done, we get all the possible cycles of Epacts, viz., 30 in number. Therefore, whatever be the cycle of Epacts in use in any century, it follows from what has been said above, that when 24 and 25 come together in the same cycle, 24 must be on the first line, viz., that embracing the *first eleven* years of the cycle, and 25 in the second line; that is to say, that both these Epacts cannot be in use together, except when 25 belongs to a Golden Number greater than XI.: and that when this condition is fulfilled, both are in use in the same cycle. It follows further, from what has been said above as to the two consecutive numbers occurring in the *same* line, that when 24 is in the first line and 25 in the second, 26 is not in the second; and therefore, when the Epact of a year is 25, by taking 26 instead of 25, the concurrence of two New Moons on the same day is avoided (1). The use of 25 for 25 will not commence until A.D. 1900 (L'art de Verif. les Dates, i., p. 2). The only harm that could have resulted from omitting this correction of 26 for 25 would be that Easter Sunday would sometimes have fallen on the same day of the month, twice in 19 years. But as this was against the rule of the Old Calendar, Clavius preferred to make the Calendar Moon wrong by a day, making it fall on the 4th of April, instead of the 5th (2).

(1). *Vid.* Clavius, pp. 120–1, and De Morgan, *l. c.*, p. 24. De Morgan remarks that this use of the double Epact 24, 25, has been often misunderstood as an *astronomical* correction, in order to make the Solar year agree better with the Lunar. This mistake is made in the great French work, L'Art de Verifier les Dates, vol. i., p. 2. "L'Epacte 25, en chiffres arabes, a été inventée pour designer en certaines années les nouvelles Lunes . . . . afin de mieux accorder l'année Lunaire avec celle du Soleil."

(2). Clavius, however, maintains (ch. x., § 9) that this anticipation of a day, so far from disturbing the Lunations, agrees better with the Moon's mean motion.

121. I now resume the observations on the Epact Calendar, which were begun in Art. 119.

3°. It will also be observed that in 7 places, viz., January 6, March 6, May 4, July 2, August 30, October 28, December 26, 25′ is written on the same line as 25. This is done in order to show that in those months 26 is not to be used for 25 (except on July 31).

4°. We also observe at the end of December the number 19 (marked with a dash, 19′) on the same line as Epact 20 (December 31). This 19 is to be used only when the Epact 19 concurs with Golden Number XIX.; a coincidence which can happen very rarely in the course of several thousand years. Thus, when we look at the Expanded Table of Epacts (Art. 123), we see that it is found only once in the whole cycle of 30 Epacts, viz., in line D. The reason why 19 is thus written opposite December 31 is the following. When Golden Number is XIX., the Embolismic month is of only 29 days. Hence, if the Epact in that year be 19, a New Moon would fall on December 2, and the Lunation would terminate on the 30th, and the next New Moon would fall on the 31st. The Epact of the year, therefore, 19, must be affixed to that day, whereas, according to the regular order, the Epact corresponding to December 31 is 20 (1).

This writing of Epact 19 on the same line with Epact 20 (the regular Epact affixed to December 31) will not occasion two New Moons to fall on the same day within the same cycle, because when Epact 19 runs with Golden Number XIX. the Epact 20 is not in use, as appears by inspection of line D in the Expanded Table of Epacts (Art. 123).

It must be observed also that by thus placing Epact 19 at December 31, the preceding Lunation ends on the 30th, and the Calendar Moon is accordingly one day old at the beginning of the next year; so that the Epact 1 of that year is formed without the usual addition of 1 to the overplus days at the end of the year. The same thing is true when Epact 18 (December 3) concurs with Golden Number XIX.: for then (as the Lunation must be one of 29 days) the Lunation will terminate on December 31, and the following Epact will be 0 or * without the addition of 1 (2).

5°. When we compare the Calendar New Moons as found by the Gregorian Epact Calendar with those found by the Old Church Calendar, we find that throughout the whole year both Calendars perfectly agree, except in 12 places in which the Epacts are more correct than the Golden Numbers, inasmuch as they give the Calendar New Moons more in accordance with the mean Moon of the heavens. But there is no difference whatsoever in the two Calendars as to the Paschal New Moons, and these, as determining Easter, are the most important. *Vid.* Clavius, pp. 123 and 377.

(1). The only other ways of meeting this difficulty would be either to write Epact 1 along with * opposite January 1, or along with 20 opposite December 31. But as Clavius shows (cap. x. § 11), both these ways are objectionable; whereas, his mode of repeating Epact 19 at December 31 has the advantage of not only being free from similar objections, but also of agreeing better with the Moon's mean place than if the Calendar New Moon were made to fall on January 1.

(2). *Vid.* Clavius, cap. x. § 14, and cap. xii. § 14. The reason why in ordinary cases of Golden Number XIX. unity must be added to the superfluous days to form the first Epact of the next cycle is, that 12 not 11 must be added to the Epact which runs with Golden Number XIX. in order to form the next Epact, which requires a corresponding addition of unity to the overplus days at the end of the year.

122. The Epacts in the Gregorian Calendar are written in a *retrograde* order. But the New Moons would have been equally correctly indicated had the Epacts been written in the natural, or direct order, *i. e.*, downwards from 1 to 30. To show this, let us take the cycle of Epacts (*D*), (Art. 116), which held from 1582 to 1699. Epact 13, *e. g.*, indicated that during that interval a New Moon fell on the 18th of March, and Epact 24 showed that on the next year of the cycle a New Moon fell on March 7. Now, if the Epacts were written in the Calendar in the *direct* order, the number of the Epact being the same as the day of the month (viz., Epact 1 corresponding to March 1; Epact 2 to March 2, and so on), we should have Epact 18 instead of 13 written opposite March 18; and the following year we should have Epact 7 (11 units *less* than 18) written opposite March 7; just as in the retrograde order 24 is *greater* by 11 than 13 of the preceding year. This mode of writing the Epacts was actually proposed by one of Gregory's mathematicians to the Calendar congregation; but the retrograde order was preferred, because the series of Epacts in the retrograde order is formed by the successive *addition* of the number 11 (as the etymology of *Epacts* seems to require), whereas in the direct order it is formed by the successive subtraction of 11 (Clavius, cap. x. § 15).

123. After these explanations we may now come to describe and point out the use of

THE EXPANDED TABLE OF EPACTS.

Equation of Epacts.							Index Letters.	Golden Numbers.																			
Index to Centuries.																											
Before Correction.		After Correction.						I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.	XIX.	
		17	18	87	88	89	C	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	25
			19	20	21	90	B	29	10	21	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	24, 25′
		22	24	91	92	93	A	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	24
			23	25	94	96	u	27	8	19	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	25
		26	27	28	95	97	t	26	7	18	29	10	21	2	13	24	5	16	27	8	19	*	11	22	3	14	24
		29	30	98	99	100	s	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	25
		31	32	33	102		r	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	28	9	20	1	12	24, 25′
		34	36	103	104	105	q	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	*	11	24
		35	37				p	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	25
		38	39	40			n	21	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	28	9	24, 25′
		41					m	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	24
		42	43	44			l	19	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	25
		45	46				k	18	29	10	21	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	24, 25′
		47	48	49			i	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	24
		50	52				h	16	27	8	19	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4	25
		51	53				g	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	*	11	22	3	24
		54	55	56			f	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	25
		57	58				e	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	28	9	20	1	24, 25′
		59	60	61			d	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	*	24
	14	62	64				c	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	25
	11	63	65				b	10	21	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	28	24, 25′
	8	66	68				a	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	24
(A.D. 320)	5	67	69				P	8	19	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	25
(A.D. 1)		70	71	72			N	7	18	29	10	21	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	24, 25′
		73	74				M	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	24
		75	76	77			H	5	16	27	8	19	*	11	22	3	14	25	6	17	28	9	20	1	12	23	25
		78	80				G	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	*	11	22	24
		79	81				F	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	25
		82	83	84			E	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	28	9	20	24, 25′
A.D.		1582	16	85	86		D	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	24

N.B.—24 or 25 is found, separately, in every line, but never both together in the same line.

25′ is never found in any line without 24 also.

This Table was originally drawn up by Lilius, and is given in its original form by Clavius, p. 9. Clavius, himself, repeats it in nearly the same form, p. 132. I have followed the arrangement given by Delambre (Astron. Mod., ii., 142), and adopted in the Encyclop. Brit., Art. "Calendar." Delambre's arrangement differs from Clavius' in two points:—1°. That Clavius commences the series of Golden Numbers with III., while Delambre begins it with I. Clavius began with III., because (as we have seen) that number was affixed to January 1 in the time of the Council of Nicæa. 2°. Lilius and Clavius commence with line P, because it is the line which agreed with the Old *perpetual* Calendar, and therefore with the date of the Council of Nice: Delambre began with C, because it was the line in use during the years from 1700 to 1899.

N.B.—The Epacts in the above Table are written in Arabic numerals to distinguish them from the Golden Numbers, which are, as usual, in Roman numerals.

124. *Explanation of the Extended Table of Epacts.*

1°. The object of this Table is to exhibit all the 30 cycles of Epacts corresponding to the 30 cycles of Golden Numbers, which arise according as the Calendar New Moons shift their places, in consequence of the Solar and Lunar Equations, through all the days, successively, of a full Lunar month, as explained already, when we were engaged with the Prayer Book Table III. (Art. 108).

2°. At the top of the Table all the Golden Numbers are written in order, beginning with Golden Number I.; and the corresponding vertical columns are crossed by 30 horizontal lines of Epacts, each line containing the 19 Epacts corresponding to the 19 Golden Numbers in use in any given century. The line of Epacts in use in any century is determined by means of the *Auxiliary Table* of the Equations of Epacts (*infr.* Arts. 167, 168).

3°. In each vertical column the 30 Epacts are written in retrograde or descending order, beginning from the top, thus producing a perpetual circulation of 30 Epacts. As Epact 1 corresponds to Golden Number I. after the correction of the Calendar in 1582 (Art. 116), the first column begins at the bottom with 1: and the rest of that line of Epacts (D) is to be written on the same horizontal row, marked D in the Table, each Epact corresponding to the Golden Number at the head of its column. In the same way the second Epact of D gives the second column, consisting of Epacts in ascending order from 12. Similarly, the third Epact (23) of D will give the whole of the third column, and so on to the end. The top row is marked C, and corresponds with (C), Art. 118. The Expanded Table must be regarded as a circle returning into itself, so that D and C are contiguous.

4°. The line C differs by 1 from the line D (as we have before seen, Art. 118), and the next, B, is again less than C; and so on.

5°. The figures in each *horizontal* row go on increasing by 11, as in the line D, and therefore *each horizontal row is a line of* EPACTS. The figures of each row, in the ascending order, exceed by unity the figure of the row immediately under it.

6°. There is one exception to the formation of the line of Epacts in passing from one column to the next by the continued addition of 11: this exception occurs in passing from column 19, Golden Number XIX., to column 1, Golden Number I., when 12 must be added instead of 11. The reason of this has been already given (Art. 118).

7°. The column of letters at the left of the Epact series consists of 30 different letters, 19 small (a, b, c, ... u), and 11 large (A, B, C, D, E, F, G, H, M, N, P). Each of these letters is called the *Index* Letter of the line of Epacts in the same horizontal row with it [1].

8°. The Epacts 24 and 25 occur together *eight* times in the course of the 30 lines of Epacts, viz., in the lines whose *indices* are E, N, b, e, k, n, r, B: and 24 always is found somewhere in the first *eleven* columns (Golden Numbers I.–XI.); while 25′ is found in the last *eight* (Golden Numbers XII.–XIX.) We have already seen (Arts. 119, 120) that when 24 and 25 are both in use in the same line of Epacts, 26 must be used for 25. In the Expanded Table, this is denoted, as in the Epact Calendar, by putting a trait over 25, thus (25′); and the margin (after Golden Number XIX.) points out the 8 lines in which 25 is thus written.

9°. The effect of the *Solar Equation* (as we know already) is to make all the New Moons in each year of the centuries (after 1600) whose centurial years are not divisible by 4 to fall one day lower or later in the several months than before; in other words, the Moon's age at the end of the year is one day less than it would have been if the intercalation had been made. Accordingly, whenever the Solar Equation takes place, the previous line of Epacts must be *diminished* by unity; and so the Epacts in the Expanded Table diminish by unity in *descending*. This diminishing of all the Epacts by 1 is equivalent to taking the next line lower down. Again, since the effect of the Lunar Equation is to make the Calendar New Moons occur one day *earlier* than before, the preceding line of Epacts must be *increased* by unity. Hence, in the centurial years 1800, 2100, 2400, &c., when the Lunar Equation alone takes place, it is necessary to *pass from one line of Epacts to the next higher* in the Expanded Table. In the centurial years in which *both Solar and Lunar Equations are made*, they *neutralize* each other, and the *line of Epacts is not changed*. The same is of course true, when neither Equation occurs.

(1). The reason why o, I, K, L, O, were omitted was to avoid a possible misconception or confusion. O, small and large, might be mistaken for 0, the cypher; I for the numeral I.; K for small k; and L for 50.—*Clavius*, cap. xi. § 2.

125. After the reformation of the Calendar, in 1582, the line of Epacts was given by D, as we have already seen (Art. 116). 1600 was continued to be a Leap-year, and therefore no Solar Equation took place. Line D, accordingly, lasted from 1582 to 1699. In 1700 the Solar Equation took place; all the Epacts were *diminished* by unity; and line C became the line of Epacts during that century (Art. 118). In 1800, *both* Solar and Lunar Equations took place; the effect of the former being to *diminish* the Epacts by unity, and of the latter, to increase them by unity, they accordingly neutralized each other, and the line C *continued* in use, and will continue until the end of 1899. In other words, C is the line of Epacts belonging to this present century. In 1900, the Solar Equation will again take place; there will be no Lunar Equation: consequently, the line of Epacts will be B. In 2000, there will be neither Solar nor Lunar Equation; so that B will still remain unchanged. In 2100, there will be *both* Solar and Lunar Equations; so that B will continue to hold good until 2199. It is easy to continue this process, which we have, in fact, already gone through (Arts. 106, 107, 118).

126. In the column under "Centuries," and on the same horizontal lines with the Indices, are written, for convenience of reference, the centuries during which the lines of Epacts denoted by the several Indices are in use.

Compare with this the Expanded Table of Golden Numbers (Art. 106). It must be observed that only the *centurial figures* of each century are written: *e. g.*, 16 (for 1600); and that in each case the century ends with the ninety-ninth year: *e. g.*, 16 denotes the 100 years from 1600 to 1699, *both inclusive.*

I have given the centuries as far as A. D. 10,000. That is to say,

The above column of centuries furnishes, in fact, a summary of what Clavius (p. 134) terms THE TABLE OF THE EQUATION OF THE EPACTS, and serves to point out what line of Epacts should be used in any century from A. D. 1600 to A. D. 10,000 (Clavius, *l. c.* p. 168). It corresponds, as may be easily seen from what has been said above, to the Equation Table II. in the Prayer Book, and to the more extended Table given in Art. 83 (*vid.* also Art. 106). Thus, *e. g.*, if we desire to know what line of Epacts should be employed during the eighteenth century, commonly so called—viz., from 1700 to 1799—we look for 17, and find that C is the line. Again, if we seek the line for the nineteenth century, we see that it also is C. If for the seventeenth century, look for 16 and find D, which also holds from 1582.

127. Hitherto we have considered this Expanded Table of Epacts only in relation to the centuries *after* the reformation of the Calendar. Clavius applies it to the *Old* Calendar also in the following manner.

At the time of the Council of Nice, in the third year of the Lunar Cycle, a New Moon fell on the 1st of January, and the Golden Number III. was accordingly affixed to January 1 in the Old Calendar (Art. 53). Hence, *if the Epacts had been then in use*, the asterisk (*) would have been affixed to that day (Art. 113); and the line of Epacts P (which corresponds to * and Golden Number III.) in the Expanded Table represents the line of Epacts, with their corresponding Golden Numbers at the time of the Nicene Council: viz.—

G. N.	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.,	I.,	II.	(P)
Epacts.	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	8	19.	

The year of the Nicene Council was 325 A. D.; but Clavius prefers to prefix P to A. D. 320, because it is a Leap-year, and therefore better adapted to the Equation Table, which proceeds by centurial years which are Bissextile. Reckoning back to the time of Christ, the Calendar New Moon (A. D. 1) fell a day later than it did in A. D. 320; because the Calendar New Moon in 320 coincided nearly with the real New Moon, which real New Moon gained a day on the Calendar Moon of 300 years before (*vid.* Clavius, cap. xi. § 4); and, consequently, the line of Epacts corresponding to A. D. 1 would be the next below P: *i. e.*, N (Clavius, xi. 4).

Now, proceeding onward from A. D. 320, the next Equation of the Moon should, in accordance with the general rule, be made in A. D. 600, and the line next above P (viz., a) should be the next line of Epacts. But inasmuch as the Calendar New Moons, and therefore the Golden Numbers, at the time of the Council of Nice, agreed *more closely* with the mean New Moons than the Paschal rules admitted of (1), Clavius deemed it better to allow an interval of time to elapse after the Nicene Council long enough to separate sufficiently the Calendar Moon from the mean Moon, and to *commence* from *that date* the Equation of the Moon every 312½ years. Now, in the year A. D. 550 (*i. e.*, about 200 years after the Nicene Council), such sufficient interval had occurred. The Golden Numbers indicated the New Moons 16 hours later than at the time of the Council of Nice. Here he places what he calls the radix or origin of the Lunar Equation. But as the secular Equations—Lunar as well as Solar—are made at the beginning of a century, P is affixed to A. D. 500, to denote that the same line of Epacts which *less correctly* denoted the Calendar New Moons at the time of the Nicene Council more correctly denoted those in the year A. D. 500. The next Lunar Equation takes place A. D. 800; and the line of Epacts whose index is a must be substituted for P. Similarly, b belongs to A. D. 1100, and c to A. D. 1400. This brings the Equation of the Moon down to the date preceding the reformation of the Calendar.

(1). We have already seen (Art. 55) why the rules of the Church in reference to Easter required that the Calendar New Moons should not coincide with, but follow at a short interval, the mean New Moon.

128. In 1582, as we have seen, 10 nominal days were dropped, in order to clear off the accumulated error arising from the excess of the Julian year over the true Tropical year. The effect of this omission was to make the Calendar New Moons fall 10 days later during the remainder of the year; so that the Epact of that year (whose Golden Number was VI.) was lowered 10 places after the omission, becoming thereby 26 instead of 6 (6 + 30 - 10 = 26). Ten lines of Epacts were passed over (descending); and thus, while c indicated the line of Epacts up to the 5th of October, 1582, the line became D during the rest of that year, as we have already seen. Thus matters stood, as regards the Epacts, after the reformation of the Calendar; and no other change took place until the year 1700, when the Solar Equation required the Epacts to be all lowered by unity, and the line to be changed from D to C. [It will be remembered, as before stated, that the Expanded Table is to be regarded as a circle, returning into itself, so that C is contiguous to D]. Although 300 years had elapsed since the Lunar Equation in 1400, yet no Lunar Equation took place in 1700. The reason was this. The actual *anticipation* of the Moon amounts (as the Gregorian correctors calculated) to one day in 312½ years. But the odd 12½ years, omitted four times, reckoning from A. D. 500 to 1700—viz., in 800, 1100, 1400, 1700—would amount to 50 years; which, added to the 50 years anticipated by making the Equation in 800 instead of 850 (550 + 300 = 850), make up 100 years. So that, had the Lunar Equation been made in 1700, the Equation would have been made 100 years sooner than it ought. Consequently, Clavius deferred the Lunar Equation to A. D. 1800; at which date, as we have already seen, the Lunar Equation took place for the first time subsequently to the reformation of the Calendar.

From what has been just said, it will be plain why Clavius, in applying the Expanded Table of Epacts to the centuries *before* 1582, assigned the Index N to A. D. 1; P to 320 and 500; a to 800; b to 1100; and c to 1400.

As the line P represents the cycle of Epacts and the Golden Numbers at the time of the Council of Nice, and as, in the Old Calendar, no change was made in the Golden Numbers until the correction in 1582, it follows that P represents the relation between the Golden Numbers and the Gregorian Epacts (the Gregorian *Epacts* denoting the Moon's age at *the beginning of a year* or the end of the preceding) during the whole time from 325 to 1582.

129. From what has been said, it is now easy to show the close connexion subsisting between Table III. of the Prayer Book and the Expanded Table of Epacts (Art. 108). In fact, the latter Table may be transformed into 30 Calendars, in which the Calendar New Moons are expressed in Golden Numbers. And in the same way, Table III. may also be expanded into 30 Calendars of Golden Numbers. These 30 Tables will,

indeed, be, properly speaking, only 30 different *Paschal Tables;* but, as has been before remarked, these Paschal Tables may each be expanded into the complete Calendar for the whole year (Art. 111). The two sets of 30 Calendars of Golden Numbers, arrived at in these two ways, will be found to be exactly identical with each other. Thus, for example, suppose we wish to form the Calendar with Golden Numbers for the period of time during which the line of Epacts denoted by Index P was in use, we have only to take into our hands the *Gregorian Perpetual Calendar* (Art. 114), and write the Golden Number III., corresponding to Epact *, opposite January 1, 31, March 1, 31, April 29, May 29, June 27, July 27, August 25, September 24, October 23, November 22, December 21. Similarly, Golden Number IV., corresponding to Epact 11, must be written opposite all the days to which Epact 11 is affixed in the Gregorian Calendar: viz., January 20, February 18, March 20, April 18, May 18, June 16, July 16, August 14, September 14, October 12, November 11, December 10. And so with all the rest of the Epacts in line P, and the Golden Numbers corresponding respectively to each. When this process is complete, we shall have the *Golden Number* Calendar corresponding to the line of Epacts P. Now, we have already seen that P is the line of Epacts belonging to the date of the Council of Nice; in other words, the Golden Number Calendar thus developed is the Old Church Calendar, as will be seen by comparing it with the latter, as set forth in Art. 53.

Table III. *begins after* the correction, and, therefore, we cannot develop from it the Old Church Calendar, except, indeed, by taking Index Figure 23, which, as we have already seen, will reproduce Old Style (Art. 106). Looking, then, at III., 23, we find April 13 for the date of the Paschal *Full* Moon; therefore, that of the New Moon was March 31, where we find Golden Number III.; and Golden Number III. is also found at January 31 and January 1. Similarly, IV., 23, gives April 2 for *Full* Moon, and March 20 for New Moon, opposite which we find Golden Number IV., and therefore IV. is also opposite January 20. In this way we shall reproduce from Table III. also the Old Paschal Table, as we developed it from the line of Epacts P.

Again, we can, in exactly the same way, derive from line D the corresponding Calendar (that in use from 1583 to 1699) in terms of the Golden Numbers. And the same Calendar can be developed, as before, from Table III., by means of the Paschal Table under Index Figure 0: and so on for the rest of the 30 Tables.

It is obvious, on comparing these results, that the 30 *Index Figures* in the Table, Art. 106, correspond to the 30 *Index Letters* in the Expanded Table of Epacts—thus:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
D	C	B	A	u	t	s	r	q	p	n	m	l	k	i
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
h	g	f	e	d	c	b	a	P	N	M	H	G	F	E.

Each corresponding pair will give the same Calendar of Golden Letters—the figures from Table III., the letters from the Expanded Table of Epacts. We shall see still more plainly the intimate connexion between these two Tables, when we come to the *Gregorian Perpetual Paschal Table*, Art. 132.

The 30 Calendars of Golden Numbers, corresponding to the 30 lines of Epacts in the Expanded Table of Epacts, are given at length by Clavius, cap. xv., pp. 285, *sq.*

130. *To Find Easter Day by means of the Gregorian Epacts.*

We can now easily see how Easter Day is found for any year A. D. (x) since the correction of the Calendar in 1582, by means of the Gregorian Epacts:—

1°. Find, from the Index to the *Expanded Table of Epacts*, the *line* of Epacts corresponding to the century to which the given year belongs.

2°. Find the Golden Number of the year by means of the formula $\left(\frac{x+1}{19}\right)_r$.

3°. From the same Expanded Table find the point of intersection of the line of Epacts (1°) with the vertical column under the Golden Number (2°). The number at that point is the Epact for the given year.

4°. Look out, in the Gregorian Calendar of Epacts (Art. 114), between the limits of March 8 and April 18 (both inclusive), for this Epact (3°) : the corresponding day of the month will be the day of the Paschal *New* Moon.

5°. Reckon 14 days from that day, inclusive, and we get the day of the Paschal *Full* Moon.

6°. Find the Sunday Letter of the given year, either from the Table, Art. 95, or the general formula, Art. 92. In the column of Calendar Letters (Table, Art. 114), take the Letter just found, which follows *next after* the day of Full Moon.

7°. The day of the month corresponding to this Letter will be Easter Day. The reason why the Letter must be the next *after* the day of Full Moon is, that Easter Day cannot fall *on* the day of Full Moon, but must be postponed to the following Sunday (Art. 45).

131. Ex. 1. Find Easter Day, A. D. 1875 (*vid.* Tables, Arts. 114 and 123).

1°. In the column of centuries in the Expanded Table of Epacts, we see that 18 has for its line of Epacts C.

2°. The Golden Number of the year is XIV.

3°. The corresponding Epact in line C is 23.

4°. In the Calendar of Epacts, within the assigned limits, this Epact 23 is found opposite March 8, which is, therefore, the day of the Paschal *New* Moon.

5°. The corresponding Full Moon falls on March 21.

6°. The Sunday Letter of the given year is C (Art. 92). This Letter belongs to March 21; consequently, we must select the *next* Sunday—namely, March 28.

7°. Therefore, Easter Day, A. D. 1875, fell on March 28.

If the given year be a Leap-year, the Sunday Letter found by the formula will be the second of the *two* Sunday Letters, as before shown. But, since Easter always falls *after* the intercalated day, the *second* Letter is the one required.

Ex. 2. Find Easter Day for the year A. D. 2176.

The line of Epacts corresponding to century 21 is B; the Golden Number XI.; the Epact 19; Sunday Letters G F. From the Epact Calendar we see that Epact 19 gives the Paschal *New* Moon on March 12, and, therefore, the *Full* Moon on March 25; and the next F after this day gives March 31 for Easter Day.

Ex. 3. Find on what day Easter will fall in A. D. 1973.

The line of Epacts is B; Golden Number XVII.; Epact 25. Therefore (Art. 120), Paschal *New* Moon falls on April 4; *Full* Moon on April 13. Sunday Letter G; consequently, Easter Day April 22.

It is obvious that, in the Old Calendar, the *Golden Number* and Sunday Letter were the necessary and sufficient data for finding Easter; so, in the Gregorian Calendar, the necessary and sufficient data are the *Epact* and the Sunday Letter. But, as both the Golden Number and the Sunday Letter are functions of the single *independent* variable, the given year (x), and as the *Epact* is deduced, as we have already intimated, and shall see more fully hereafter (Art. 137), from the Golden Number, it follows that, both in the Old and New Calendars, there is but one necessary and sufficient datum for the determination of Easter—viz., the year A. D.

132. We have seen (Art. 60) that, in the *Old Calendar*, so much of it as related to the finding of Easter Day was separated from the rest, and formed what was called the *Perpetual Paschal Table* (Art. 60); and (Arts. 105, 106) that, after the correction, similar *temporary* Paschal Tables might be, and were, constructed for finding Easter Day, as before, by means of the Golden Numbers: for example, the two temporary Paschal Tables in our Prayer Book. Similarly, so much of the *Gregorian Epact Calendar* has been extracted from the rest as is contained between the limits of the Paschal *Full* Moons (March 21 to April 18); and this Table forms what is called *The Perpetual Gregorian Paschal Table.*

In this Table the Epacts denote, not, as in the general Epact Calendar, the Paschal *New* Moons (March 8 to April 5), but the Paschal Full Moons (March 21 to April 18); which is effected by simply moving down all the Epacts within the former limits *thirteen* places; just as in the Old Perpetual Paschal Table the Golden Numbers were all moved down 13 places, in order to denote *Full* Moons. Thus, Epact 23, which is affixed to March 8, is moved down to March 21; and so the following Epacts, as far as 25, 24 (April 5), are also moved down, the last being opposite April 18. Thus we get the

PERPETUAL GREGORIAN PASCHAL TABLE.

Days of Month.	O. S. G. N. Full Moon.	N. S. Epact Full Moon.	Cal. Letters.	Days of Month.	O. S. G. N. Full Moon.	N. S. Epact Full Moon.	Cal. Letters
March 21	XVI.	23	C	April 8		5	G
„ 22	V.	22	D	„ 9	XVII.	4	A
„ 23		21	E	„ 10	VI.	3	B
„ 24	XIII.	20	F	„ 11		2	C
„ 25	II.	19	G	„ 12	XIV.	1	D
„ 26		18	A	„ 13	III.	*	E
„ 27	X.	17	B	„ 14		29	F
„ 28		16	C	„ 15	XI.	28	G
„ 29	XVIII.	15	D	„ 16		27	A
„ 30	VII.	14	E	„ 17	XIX.	25, 26	B
„ 31		13	F	„ 18	VIII.	25, 24	C
April 1	XV.	12	G	„ 19			D
„ 2	IV.	11	A	„ 20			E
„ 3		10	B	„ 21			F
„ 4	XII.	9	C	„ 22			G
„ 5	I.	8	D	„ 23			A
„ 6		7	E	„ 24			B
„ 7	IX.	6	F	„ 25			C

The second column (Golden Numbers) is, strictly speaking, no part of this Table. They belong to the Old Calendar, and denote the *Full* Moons as denoted by the Golden Numbers, previous to the correction, 1582 (Art. 60). They are added here merely for the sake of more easy comparison of the New and Old Style Easters. The Epact column will show the former, and the Golden Number column will show the latter, Easter Day, which is still observed in the countries that have not adopted the Gregorian reckoning.

The mode of using this Epact Table, to find the New Style Easter, is very obvious, being, in fact, perfectly similar to the mode of finding the Old Style Easter Day by the Old Paschal Table (Art. 60).

The Epact of the year being found, as before (Art. 130, 1°, 2°, 3°), and the Sunday Letter being also found, look for the Epact in this Table, and for the Sunday Letter next *after* that Epact. The corresponding day of the month will be Easter Day.

Ex. 1. Find Easter Day, A. D. 2000.

The Epact is 24; Sunday Letters B A. By the Table, 24 belongs to April 18; and the next following A to April 23; which is, therefore, the Easter Day required.

Ex. 2. Find Easter Day, A. D. 2182.

The Epact is 25′; and Sunday Letter F. Therefore the required date is April 21.

Ex. 3. Find Easter Day, A. D. 2221.

The Epact is 5; Sunday Letter G; Epact 5 belongs to April 8, whose Letter also is G. Hence we must take the following G, which gives April 15 for the required Easter Day.

If we desire the Old Style Easter—that is, the date of Easter Day, had there been no correction of the Calendar—for the years above given, we find that the Golden Numbers for the years 2000, 2182, 2221, respectively, are VI., XVII., XVIII. The Sunday Letters are generally the same as those of New Style, but must be looked for in the Table, Art. 29, or calculated by the formula, Art. 32. In the present cases, the Sunday Letters are, respectively, C B, E, F. Hence, by the above Table, the corresponding Easters fall on April 17, April 14, and March 31.

133. The following mode of finding Easter Day when the Epact and Sunday Letter are known is very simple.

Looking at the Epact Calendar (Art. 114), we see that in March the Epact which indicates the New Moon, added to the day of the month, is always = 31. If we add 13, we see that the date of the Paschal Full Moon, if it falls in March, is = 13 + 31 − Epact = 44 − Epact. But if the Full Moon falls in April, its date will be 43 − Epact, since *as far as April* 5 the day of the month + Epact = 30, and the Paschal New Moon cannot fall later than April 5. Further, as no Paschal New Moon can fall before March 8, corresponding to Epact 23, it follows that in March the Epact of the year cannot be greater than 23.

1. *If, then, the Epact be* < 24, find it in March and take the date 44 − *Epact*, which will give the Paschal Full Moon, it being understood that March 32 = April 1, March 33 = April 2, and so on.

2. *If the Epact be* > 24, look for it in April, and take the date 43 − *Epact in April;* this will give the Paschal Full Moon.

3. As no Paschal New Moon can fall after April 5, the Epact of which is double (24 and 25), and no Paschal Full Moon can fall after April 18, it follows that if the *Epact*

of the year be 24, it must be changed to 25: because 43 - 24 would give the date the 19th of April.

4. We have also seen that if the Epact of the year be 25, and at the same time the Golden Number be > 11, we must change 25 into 26.

5. Easter Day is the *Sunday* after the date thus found for the Paschal Full Moon; and that Sunday is found as before when we *know the Sunday Letter*.

Ex. 1. Find Easter Day, A. D. 1848, given that the Epact is 25, and Sunday Letter A.

As Epact is > 24, we must look for 25 in April. But as the Golden Number, being VI., is less than XI., we need not take 26 instead of 25. Hence, date of Paschal Full Moon = April (43 - 25) = April 18; and the Letter A is found opposite the 23rd. Therefore Easter Day falls on April 23.

Ex. 2. Find Easter Day for the year 1954.

The Epact is 25, and Sunday Letter C; Golden Number XVII. As the Golden Number > XI., 25 must be changed to 26: therefore, 43 - 26 = 17 April is the day of Paschal Full Moon; and the Letter C is found opposite the 18th, therefore the required day is April 18.

Ex. 3. Find Easter Day for the year 1818.

Epact is 23, and Sunday Letter D: accordingly, 44 - 23 = 21st of March is the day of Paschal Full Moon; and D is found opposite the 22nd, consequently, 22nd of March is Easter Day (earliest possible).

Ex. 4. Find Easter Day for A. D. 2258.

The Epact of the year is 24, and the Sunday Letter C: 24 must first be changed into 25 (*supra*), and then we have 43 - 25 = 18 April, as the date of the Paschal Full Moon; and the Letter C is found opposite April 25, therefore Easter Day falls on April 25 (the latest possible).

Ex. 5. The same is true for A. D. 4900.

134. We have given (Art. 62) a *Table to find Easter for ever*, according to the old unreformed Church Calendar. A similar Table has been constructed by Clavius (p. 38) for the *Gregorian* Calendar, in which the Epacts take the place of the Golden Numbers. The Table shows all the days on which Easter can possibly fall, according to the possible combinations of the 31 Epacts (25′ being counted as distinct from 25) and seven Sunday Letters. The construction of the Table is obvious on inspection, and from the explanations appended to it.

The Gregorian Paschal Table,

Showing the days on which Easter can fall according to the possible combinations of the Epacts and Sunday Letters.

Almanac.	Dom. Lett.	Cycle of Epacts.	Easter Day.
1		23	22 M.
8		22 21 20 19 18 17 16	29 M.
15	D	15 14 13 12 11 10 9	5 A.
22		8 7 6 5 4 3 2	12 A.
29		1 * 29 28 27 26 25′ 25 24	19 A.
2		23 22	23 M.
9		21 20 19 18 17 16 15	30 M.
16	E	14 13 12 11 10 9 8	6 A.
23		7 6 5 4 3 2 1	13 A.
30		* 29 28 27 26 25′ 25 24	20 A.
3		23 22 21	24 M.
10		20 19 18 17 16 15 14	31 M.
17	F	13 12 11 10 9 8 7	7 A.
24		6 5 4 3 2 1 *	14 A.
31		29 28 27 26 25′ 25 24	21 A.
4		23 22 21 20	25 M.
11		19 18 17 16 15 14 13	1 A.
18	G	12 11 10 9 8 7 6	8 A.
25		5 4 3 2 1 * 29	15 A.
32		28 27 26 25′ 25 24	22 A.
5		23 22 21 20 19	26 M.
12		18 17 16 15 14 13 12	2 A.
19	A	11 10 9 8 7 6 5	9 A.
26		4 3 2 1 * 29 28	16 A.
33		27 26 25′ 25 24	23 A.
6		23 22 21 20 19 18	27 M.
13		17 16 15 14 13 12 11	3 A.
20	B	10 9 8 7 6 5 4	10 A.
27		3 2 1 * 29 28 27	17 A.
34		26 25′ 25 24	24 A.
7		23 22 21 20 19 18 17	28 M.
14		16 15 14 13 12 11 10	4 A.
21	C	9 8 7 6 5 4 3	11 A.
28		2 1 * 29 28 27 26 25′	18 A.
35		25 24	25 A.

Note.—Whenever the Epact is 25′, which can only be when the Golden Number $>$ 11, we *must* change 25′ to 26.

Whenever the Epact is 24 we *may* change it into 25; but with Letter C this change is *necessary*.

The Column entitled "Almanac" will be explained further down (Art. 165).

There are 217 possible combinations of the seven Dominical Letters with the 31 Epacts (25′ being counted as one); viz., 31 for each separate Letter. Each of these seven different groups of thirty-one gives five different days on which Easter may fall; the whole seven groups giving the entire thirty-five.

Only *one* Epact (23), combined with Letter D, will give the earliest possible Easter, March 22nd.

Only *two* Epacts (23, 24), combined with Letter E, will give Easter on March 23rd.

Only *three* Epacts (23, 22, 21), combined with Letter F, will give Easter on 24th of March. And so on up to the 28th of March, when any one of the seven Epacts 23 . . . 17, combined with Letter C, will give that day.

On the other hand, the other extreme limit of Easter, viz., April 25, will be given by either of the *two* Epacts 25 or 24, combined with Letter C. This accounts for the much more frequent occurrence of this extreme limit, than of the other (March 22). *Vid.* Art. 71.

Again, either of the four Epacts 26, 25′, 25, 24, combined with Letter B, will give Easter on April 24: and so on to April 19, when any of the nine Epacts, 1, *, 29, 28, 27, 26, 25′, 25, 24, combined with Letter A, will give that day as Easter Day.

For *each* of the other days contained between March 29 and April 17, both inclusive, there are 7 Epacts with a corresponding Letter, which will give Easter on one of those days. In the case of April 18, *eight* Epacts, combined with Letter C, will give Easter on that day; and the same is true of April 20, when the Sunday Letter is E. Hence, it is easy to see why Easter ought to fall most frequently on one of the days included between March 29 and April 20.

The use of this Table is obvious.

Ex. 1. Given the Epact 23 and Sunday Letter C, find date of Easter Day.

Inspection shows that it falls on March 28. This is true, for instance, of 1875.

135. Such is the mode of finding Easter by the Gregorian method of Epacts, by means of Tables.

Clavius relied almost exclusively on the Tables, for determining Easter; and his great work, so often referred to, contains a number of most elaborate Tables for this purpose. The famous German mathematician, Gauss, was the first, or among the first, who substituted algebraic forms for Clavius' Tables, though his very ingenious rule (Art. 149) does not wholly dispense with the use of a Table. To Delambre is due the credit of being the first to give a purely analytical solution of the problem to find Easter Day. The following is substantially his method of solution ("Astron. Mod.," i. pp. 8, *sq.*)

The first step in the process is to investigate a *Formula for finding the Epact for any* *year.* Now, we have seen (Art. 117) that, *after the correction in* 1582, and before the Secular Equations began, the relation between the Epact and the Golden Number (N) was given by the formula

$$\text{Epact} = \left(\frac{N + 10\,(N-1)}{30}\right)_r = \epsilon_1.$$

Let ⊙, as before, denote the Solar Equation, and ☽ the Lunar; and let ϵ be the corrected Epact for *any* year A. D.; then, as the effect of the Solar Equation is to *diminish* the Epact (ϵ_1), and that of the Lunar Equation to *increase* it, we shall have, for *any* year A. D.,

$$\epsilon = \epsilon_1 - \odot + ☽. \qquad (1)$$

But (by Art. 87) $\quad \odot = (\sigma - 16) - \left(\frac{\sigma - 16}{4}\right)_w = \left(\frac{3\,(\sigma - 15)}{4}\right)_w$; (Art. 87, Note 2, (*f*)):

and (Art. 88),

$$☽ = \left(\frac{\sigma - 15 - a}{3}\right)_w.$$

Substitute these values of ⊙ and ☽ in (1), and we get for the *corrected Epact*

$$\epsilon = \left(\frac{N + 10\,(N-1)}{30}\right)_r - (\sigma - 16) + \left(\frac{\sigma - 16}{4}\right)_w + \left(\frac{\sigma - 15 - a}{3}\right)_w. \qquad (2)$$

This formula contains the whole of the *Expanded Table of Epacts.*

Since N and σ are both functions of x, the given year, the Epact is a function of x only. The three last terms depend on σ only, and therefore remain unchanged for a century.

Since $a = \left(\frac{\sigma - 17}{25}\right)_w$, (Art. 88), its value will be zero until $\sigma - 17 = 25$; that is to say, until A. D. 4200; and up to that date the third term may be written $\left(\frac{\sigma - 15}{3}\right)_w$. Moreover, $\left(\frac{\sigma - 15}{3}\right)$ will not itself be a whole number till $\sigma = 18$; in other words, the Lunar Equation itself did not come into the calculation of the Epact until A. D. 1800; so that up to that date, the expression for the Epact was

$$\epsilon = \left(\frac{N + 10\,(N-1)}{30}\right)_r - (\sigma - 16) + \left(\frac{\sigma - 16}{4}\right)_w.$$

Again, $\frac{\sigma - 16}{4}$ will not be a whole number until $\sigma = 20$; therefore, for any year after 1799 and before 2000, the value of the Epact is given by the expression

$$\epsilon = \left(\frac{N + 10\,(N-1)}{30}\right)_r - (\sigma - 16) + \left(\frac{\sigma - 15}{3}\right)_w.$$

For the present century, as $\sigma = 18$,

$$\epsilon = \left(\frac{N + 10\,(N-1)}{30}\right)_r - 1 = \left(\frac{(N-1)\,11}{30}\right)_r,$$

as we found before (Art. 116, 3°).

136. From the general expression (2), Art. 135, results the following *Rule for finding the Gregorian Epact for any year* (x) A. D.

1°. Find the Golden Number of the given year: $\left(N = \left(\frac{x+1}{19}\right)_r\right)$.

2°. To N add ten times the next lower number; divide by 30, and keep the remainder: $\left(\frac{N + 10(N-1)}{30}\right)_r$.

3°. From the *centurial* figures of the year (assumed to be not less than 16) deduct 16: $(\sigma - 16)$.

4°. Take the quotient of 3° divided by 4, neglecting fractions: $\left(\frac{\sigma - 16}{4}\right)_w$.

5°. From the centurial figures of the year deduct 17; divide by 25, and keep the quotient: $\left(\frac{\sigma - 17}{25}\right)_w$.

6°. From the centurial figures subtract 15 and 5°; divide by 3, and keep the quotient: $\left(\frac{\sigma - 15 - a}{3}\right)_w$.

7°. Add together 2°, 4°, 6°, and subtract 3°. The difference will be the Epact.

NOTE.—If the result be *negative*, subtract it from 30, and the remainder is the Epact; if 0, change it into 30.

If the result be 24, change it into 25, and if 25, change it into 26, whenever the Golden Number is greater than XI. (Art. 119).

Ex. 1. Find the Epact for the year 4210.

1°. G. N. $= \left(\frac{4211}{19}\right)_r =$ XII.

2°. $\left(\frac{12 + 10.11}{30}\right)_r = 2.$

3°. $(42 - 16) = 26.$

4°. $\left(\frac{42-16}{4}\right)_w = 6.$

5°. $\left(\frac{42-17}{25}\right)_w = 1.$

6°. $\left(\frac{42-15-1}{3}\right)_w = 8.$

7°. $2° + 4° + 6° - 3° = 16 - 26 = -10 = 20$ (*i. e.*, $30 - 10$).

Hence the required Epact is 20.

Ex. 2. Find the Epact for the year 1954.

1°. G. N. = XVII.; 2° = 27; 3° = 3; 4° = 0; 5° = 0; 6° = 1.

Hence, $2° + 4° + 6° - 3° = 25$, the Epact, which, as G. N. > XI., must be changed to 26 [1].

(1). There may be, of course, other formulæ for finding the Epact, different from that given in the text. Delambre, *l. c.*, pp. 51–57, gives the following, by Ciccolini:—

$$\epsilon = \left\{\frac{11\,N - \left(\frac{3\sigma - 5}{4}\right)_w + \left(\frac{8\sigma - 112}{25}\right)_w}{30}\right\}_r. \qquad (a)$$

This is at once deducible from equation (2), Art. 135, which may be written thus—

$$\epsilon = \left(\frac{11\,N - 10 - \odot + \rlap{)}}{30}\right)_r, \qquad (b)$$

where, as before, $\odot$ stands for the Solar, and ☽ for the Lunar, Equation. Now, by Art. 87, Note 2, equation (f),

$$\odot = \left(\frac{3(\sigma - 15)}{4}\right)_w = \left(\frac{3\sigma - 5}{4}\right)_w - 10;$$

and by Art. 88, Note,

$$☽ = \left(\frac{8\sigma + 13}{25}\right)_w - 5 = \left(\frac{8\sigma - 112}{25}\right)_w.$$

By substitution, we at once get the required value for ϵ.

Another expression is the following:—

$$\left\{\frac{11\,N - \left(\frac{17\left(\frac{\sigma}{4}\right)_r + \left(\frac{\sigma}{4}\right)_w \times 43 + 86}{25}\right)_w}{30}\right\}_r. \qquad (c)$$

Taking equation (b), we see that

$$\odot = \sigma - 16 - \left(\frac{\sigma - 16}{4}\right)_w = \sigma - \left(\frac{\sigma}{4}\right)_w - 12$$

$$☽ = \left(\frac{8\sigma - 112}{25}\right)_w;$$

and, therefore,

$$\epsilon = \left(\frac{11N - \left\{\sigma - \left(\frac{\sigma}{4}\right)_w - 2 - \left(\frac{8\sigma - 112}{25}\right)_w\right\}}{30}\right)_r.$$

By equation (e') of Art. 87, Note 2, we see that

$$\sigma - \left(\frac{\sigma}{4}\right)_w - 2 - \left(\frac{8\sigma - 112}{25}\right)_w$$

$$= \left(\frac{25\left(\sigma - \left(\frac{\sigma}{4}\right)_w - 2\right) - 8\sigma + 112 - 1}{25}\right)_w + 1$$

$$= \left(\frac{17\sigma - 25\left(\frac{\sigma}{4}\right)_w + 61}{25}\right)_w + 1$$

$$= \left(\frac{17\sigma - 25\left(\frac{\sigma}{4}\right)_w + 86}{25}\right)_w;$$

or, since $\sigma = 4\left(\frac{\sigma}{4}\right)_w + \left(\frac{\sigma}{4}\right)_r,$

the above expression $= \left(\frac{17\left(\frac{\sigma}{4}\right)_r + 43\left(\frac{\sigma}{4}\right)_w + 86}{25}\right)_w.$

By substitution, we get equation (e), written above.

137. *Having found the Epact for any year, we now proceed to show how Easter Day may be determined from it, analytically.*

The problem contains but one independent variable—viz., the year, x; as the Golden Number is a function of x, and so also the Sunday Letter. Consequently, when x is given, we have the only datum necessary for finding Easter.

Delambre's solution is substantially the following (Astron. Mod., i., p. 17). We have already employed it in Art. 64, but we repeat it here.

The *earliest* day on which Easter can fall is March 22 (Art. 45): the Calendar Letter, in this case, is D; and therefore the Sunday Letter has for its number 4, in the scale

A	B	C	D	E	F	G
1	2	3	4	5	6	7

(Art. 33). In this case, Sunday is the fifteenth day of the Moon; therefore, the day of the New Moon is March 8 (22 – 14), whose Calendar Letter is also D = 4. In this same case, the Epact is 23, as this is the Epact opposite March 8 in the Epact Calendar. Hence it follows that Easter Day will fall on March 22 (its earliest possible occurrence), when the Epact is 23 and Sunday Letter D. These are the necessary and sufficient conditions. The latest day on which Easter can fall (as we have already seen, Art. 45) is April 25, Epact 24, Sunday Letter C.

This being premised, the GENERAL RULE is thus found:—

Easter Day cannot fall earlier than March 22, or later than April 25.

In the first case, $D + \epsilon = 4 + 23 = 27$;
In the second case, $C + \epsilon = 3 + 24 = 27$.

Let P denote the day of March on which the fifteenth day of the Paschal Moon falls, and which would be Easter Day if the fifteenth of the Moon fell on Sunday [N. B.—April is to be considered as a continuation of March: *e. g.*, April 10 = March 41].

Let π denote the day of March on which Easter Day actually falls in the given year.

Let L be the *number* of the Sunday Letter of the year, in the above scale; and λ the number of the Calendar Letter corresponding to P.

The equation of the problem is obviously

$$\pi = P + L - \lambda. \quad (1)$$

L is found when the year is given. P and λ are found from the Epact (ϵ), as follows:—

Suppose Easter to fall on March 22, we have $P = 22$, and $\epsilon = 23$; and, therefore,

$$P + \epsilon = 45, \quad \text{or } P = 45 - \epsilon. \quad (2)$$

Formula (2) is true, not merely for the particular case supposed, but *in general;* because, in consequence of the retrograde order of the Epacts in the Calendar, as fast as P increases, ϵ diminishes; thus, when P becomes $22 + 1$, ϵ becomes $23 - 1$; and, in general, when P becomes $22 + y$, ϵ simultaneously becomes $23 - y$; so that their sum is constant, and $= 45$.

The only uncertainty that can arise is with reference to the double Epacts prefixed to April 4 and 5. But we have already seen (Art. 119) that, when the Epact of the year is 25, and the Golden Number > XI., we must use 26, or the day opposite 26 (April 4), as the day of the Paschal New Moon.

We have still to find λ.

In the extreme case of $P = 22$, λ is $D = 4$; and $\epsilon = 23$. Hence,

$$\lambda + \epsilon = 4 + 23 = 27, \quad \text{and } \lambda = 27 - \epsilon.$$

But this expression is general for all cases of λ; because λ increases as P increases, and ϵ decreases by the same number (as above). Hence

$$\lambda + y + \epsilon - y = 27,$$

and, therefore, in all cases,

$$\lambda = 27 - \epsilon.$$

As λ cannot exceed 7, this must be written

$$\lambda = \left(\frac{27 - \epsilon}{7}\right)_r. \tag{3}$$

Thus, then, knowing P, L, and ϵ, we find the value of π from formula (1).

P cannot be less than 22; therefore the *minimum* value of $45 - \epsilon$ is 22; or $\epsilon = 23$. If $\epsilon > 23$, we must add 30 to the constant, 45, in (2); and, in the same way, we must, in (3), add 30 to the constant, 27: because λ is the Letter corresponding to P. Hence the Rule, that, whenever $\epsilon > 23$, the formulas

$$\left.\begin{array}{ll} (2) & 45 - \epsilon \\ (3) & \left(\frac{27 - \epsilon}{7}\right)_r \end{array}\right\} \text{ must be changed to } \left\{\begin{array}{l} 75 - \epsilon \\ \left(\frac{57 - \epsilon}{7}\right)_r \end{array}\right\} \tag{4}$$

This completes the solution of the problem.

As the Epact 23 is affixed to March 8, the earliest day on which the Paschal New Moon can fall, and as the Epacts proceed in retrograde order, it follows that when the Epact is *greater* than 23 the Paschal Moon cannot fall in March, and therefore Easter must fall in April.

If, on the other hand, the Epact is *less* than 24, the Paschal New Moon cannot fall in April; and, therefore, Easter cannot fall in that month, because the last day in April on which the Paschal New Moon can fall is the 5th, to which corresponds the double Epact 25, 24.

It must also be remembered, as before mentioned, that in calculating Easter for the different years of any century a certain number of the items of calculation are common to the *whole* century, while some belong exclusively to the particular years. The *constant* elements are those which involve only the centurial figure, σ, while the *variable* ones are those into which x enters: viz., ϵ and L.

138. Having already given (Art. 136) the arithmetical Rule for finding the *Epact* of any year, I now proceed to add to it the remaining steps necessary for the determination of Easter Day, as derived from the investigation of the preceding Article. The first step is to find the *Sunday Letter of the year*.

The formula for this is given in Art. 92 (4), which is translated as follows:—

Sunday Letter.

8°. Add 1 to the given year: $(x+1)$.

9°. Take the quotient of the given year divided by 4, neglecting the remainder: $\left(\frac{x}{4}\right)_w$.

10°. Subtract 3° from this sum; and add 4°.

11°. Divide the result so obtained by 7, and keep the remainder: $\left(\frac{R}{7}\right)_r$.

12°. Subtract this remainder from 7: $L = 7 - \left(\frac{R}{7}\right)_r$.

We have now two cases to consider.

I. *When the Epact is* 23 *or less.*

13°. Subtract the Epact from 45: $(P = 45 - \epsilon)$.

14°. Subtract the Epact from 27; and divide by 7, keeping the remainder, or 7, if there be no remainder: $\lambda = \left(\frac{27-\epsilon}{7}\right)_r$.

II. *When the Epact is greater than* 23.

13°. Subtract the Epact from 75: $(P = 75 - \epsilon)$.

14°. Subtract the Epact from 57; divide by 7, and keep the remainder, or 7, if there be no remainder: $\lambda = \left(\frac{57-\epsilon}{7}\right)_r$.

15°. To 13° add 12°, and subtract 14°; if 14° > 12°, add 7: $(P + L - \lambda)$.

16°. The result is the day of March on which Easter Day falls; if more than 31, subtract 31, and the remainder is the day of April on which Easter falls: $(\pi = P + (L - \lambda))$.

The following are the formulas necessary and sufficient for finding Easter Day analytically, for any year x:—

$$N = \left(\frac{x+1}{19}\right)_r. \tag{1}$$

$$\epsilon = \left(\frac{N + 10(N-1)}{30}\right)_r - (\sigma - 16) + \left(\frac{\sigma - 16}{4}\right)_w + \left(\frac{\sigma - 15 - a}{3}\right)_w, \tag{2}$$

$$\text{where } a = \left(\frac{\sigma - 17}{25}\right)_w.$$

$$L = 7 - \left\{\frac{(x+1) + \left(\frac{x}{4}\right)_w - (\sigma - 16) + \left(\frac{\sigma-16}{4}\right)_w}{7}\right\}_r. \tag{3}$$

$$\left.\begin{aligned} P &= 45 - \epsilon \\ \lambda &= \left(\frac{27-\epsilon}{7}\right)_r \end{aligned}\right\}, \text{ when } \epsilon \text{ is 23 or less (}^1\text{).} \tag{4}$$

$$\left.\begin{aligned} P &= 75 - \epsilon \\ \lambda &= \left(\frac{57 - \epsilon}{7}\right)_r \end{aligned}\right\}, \text{ when } \epsilon \text{ is greater than } 23. \qquad (5)$$

$$L - \lambda \text{ (it may be = 0, but cannot be negative: and when } L < \lambda, \text{ add } 7). \qquad (6)$$

$$\pi = P + L - \lambda. \qquad (7)$$

(1). Both these equations, (4) and (5), can be included in the single one:

$$P = 22 + \left(\frac{30 + 23 - \epsilon}{30}\right)_r, \quad \lambda = \left(\frac{4 + \left(\frac{30 + 23 - \epsilon}{30}\right)_r}{7}\right)_r.$$

139. I shall now give some examples to illustrate the application of the above solution.

Ex. 1. Find Easter Day, A. D. 4900.

We must find the *Epact* from Art. 135, (2). In the given case,

$$\sigma = 49\,;\ \sigma - 16 = 33\,;\ \sigma - 15 = 34\,;\ \sigma - 17 = 32\,;\ a = 1\,;$$

$$\left(\frac{\sigma - 15 - a}{3}\right)_w = 11\,;\ \left(\frac{\sigma - 16}{4}\right)_w = 8 \quad \therefore \left(\frac{\sigma - 16}{4}\right)_w - (\sigma - 16) = -25.$$

These, which are the *Secular* terms of the equation (2), will be the same for the whole century 49.

The Golden Number $(N) = \left(\frac{4901}{19}\right)_r = \text{XVIII}.$

Hence, $$\left(\frac{N + 10\,(N - 1)}{30}\right)_r = 8 = 30 + 8 = 38.$$

Hence we get for the Epact,

$$\epsilon = 38 - 25 + 11 = 24.$$

We have next to find the Sunday Letter, L. For this purpose, we must take the expression for L given in equation (3) of last Article. Substituting in that expression the values just found for $\sigma - 16$ and $\left(\frac{\sigma - 16}{4}\right)_w$, and putting 4900 for x, we get $R = 6101$, and $\left(\frac{R}{7}\right)_r = 4$; therefore $L = 7 - 4 = 3$.

As $\epsilon > 23$, we must, to find P and λ, take the values given by (5); that is to say,

$$P = 75 - 24 = 51,$$

$$\lambda = \left(\frac{57 - 24}{7}\right)_r = 5,$$

$$\therefore\ L - \lambda = 3 - 5 = -2 = 7 - 2 = 5.$$

And, finally, $\pi = P + L - \lambda = 51 + 5 = 56$ of March = April 25.

If, in the above calculation of Easter, we were to use Epact 25, instead of 24, we should get $P = 50$; $\lambda = 4$; and, therefore, $\pi = 50 + 6 = 56$ of March, as before.

We see, then, that using 25 for 24 gives λ and P each less by a unit than when 24 is used; but, on the other hand, $L - \lambda$ is greater by a unit, and consequently the result is the same in both cases.

We shall show more fully, farther down (Art. 140), when 24 *may* be changed into 25, and when it *must*.

Ex. 2. Find Easter Day for the year A. D. 1954.

We have already found (Art. 136) the Epact for that year to be 25, which, as Golden Number (XVII.) is > XI., must be changed to 26. We have, then,

$$P = 75 - 26 = 49;\ \lambda = \left(\frac{57 - 26}{7}\right)_r = 3;\ L\ (\text{Art. } 138) = 3 = \text{C};\ L - \lambda = 0:$$

therefore $P = 49 + 0 = 49$ of March, or April 18, Easter Day.

Ex. 3. Find Easter Day for the year 3860.

$$N = 4;\ \left(\frac{N + 10\,(N - 1)}{30}\right)_r = 4;\ -(\sigma - 16) = -22;\ \left(\frac{\sigma - 16}{4}\right)_w = 5;\ \left(\frac{\sigma - 15}{3}\right)_w (a = 0) = 7.$$

Hence, $$\epsilon = 4 + 5 + 7 - 22 = -6 = 30 - 6 = 24.$$

$$\therefore\ P = 75 - 24 = 51;\ \lambda = \left(\frac{57 - 24}{7}\right)_r = 5;\ L = 7 = \text{G};\ L - \lambda = 2;$$

$$\therefore\ \pi = 51 + 2 = 53 \text{ of March, or 22 of } \textit{April}, \text{ Easter Day.}$$

In Clavius, p. 514, this is, by mistake, given as *March* 22. The correct day (April 22) is given by him under A. D. 3917 and 4012, which years have the same Epact (24) and Sunday Letter (G) as 3860; and therefore all three must have the same date for Easter Day.

140. I add to the examples above given a few more, tabulating the particular results leading to the final conclusion, the references 1°, 2°, &c., being to the steps enumerated in Arts. 136, 138.

Given Year x.		1853	1876	2018	4686	2285	1886
Epact.	1°	XI.	XV.	V.	XIII.	VI.	VI.
	2°	21	5	15	13	26	26
	3°	2	2	4	30	6	2
	4°	0	0	1	7	1	0
	5°	0	0	0	1	0	0
	6°	1	1	1	10	2	1
ϵ	7°	20	4	13	$0 = 30$	23	25
Sunday Letter.	8°	1854	1877	2019	4687	2286	1887
	9°	463	469	504	1171	571	471
	10°	2315	2344	2520	5835	2852	2356
	11°	5	6	0	4	3	4
L	12°	2	1	7	3	4	3
P	13°	25	41	32	45	22	50
λ	14°	7	2	7	6	4	4
$P+L-\lambda$	15°	27	47	32	49	22	56
Easter Day,		March 27	April 16	April 1	April 18	March 22	April 25

In the two last columns we have examples of the earliest and latest possible Easters, respectively; the first having Epact 23 and Sunday Letter D; the latter, Epact 25 and Sunday Letter C.

For Easter on March 22, the necessary and sufficient conditions are Sunday Letter D and Epact 23.

For Easter on April 25, the necessary and sufficient conditions are Sunday Letter C, and Epacts 25 *or* 24. Thus, for instance, in the years 1943, 2038, 2190, in which Easter falls on April 25, the Epact is 24.

141. We have seen in Ex. 1, Art. 139, that 24 may be changed into 25, without altering the result. It *may* be *always* so changed; but it *must* be changed, when the

Sunday Letter is D. Again, 25 *may always* be changed into 26; and *must* be, when the Golden Number > 11, and the Sunday Letter is C.

That the Epact 24 *may always* be changed into 25, and 25 into 26, is obvious from this, that in each case P and λ are lessened by a unit; but, on the other hand, $L - \lambda$ is increased by a unit; consequently, π $(= P + L - \lambda)$ remains unchanged.

That 24 *must* be changed into 25, when the Sunday Letter is D is easily shown, thus. Art. 139 shows that

$$P = 75 - 24 = 51$$

$$\lambda = \left(\frac{57 - 24}{7}\right)_r = 5.$$

But in this case $L = \text{D} = 4$; $\therefore L - \lambda = 4 - 5 = -1 = 6$. Consequently, $\pi = 51 + 6 = 57$ = April 26, which is *beyond* the Easter limits, and must be rejected. Changing, therefore, to 25, we get $P = 50$; $\lambda = 4$; $\therefore L - \lambda = 0$. Hence, $\pi = 50 - 0 =$ April 19, Easter Day. This change will be required in 1981 (*vid. infr.* Art. 144).

That 25 *must* be changed into 26 when Sunday Letter is C may be proved in the same way:—

$$\left.\begin{aligned} P &= 75 - 25 = 50 \\ \lambda &= \left(\frac{57 - 25}{7}\right)_r = 4 \end{aligned}\right\} : \text{therefore, } L - \lambda = 3 - 4 = -1 = 6;$$

$$\therefore \pi = 50 + 6 = 56 = \text{April 25}.$$

But when G. N. > 11, the Epact 24 is in use along with 25 (Art. 119); and Epact 24 and Sunday Letter C will also give April 25 for Easter Day. Now, two Easters on the same day in the same cycle is not allowed. But by changing 25 to 26, we get $P = 49$; $\lambda = 3$; $\therefore L - \lambda = 3 - 3 = 0$: consequently

$$\pi = 49 + 0 = \text{April 18}.$$

This will take place, for example, in A.D. 1954, 2106.

142. The above formulas (Arts. 136, 138) for finding Easter Day are, of course, applicable, with the proper modifications, to the Old, or Julian, Calendar also. In fact, the work has been already done for the latter, Art. 64. The only changes required in the Gregorian formulas are in the case of the *Epact* and the Sunday Letter. In the former we must omit the change made in 1582 by the dropping of the ten days, and the *Secular corrections* subsequently; and in finding the Sunday Letter, the Secular correction must also be omitted. The rest of the process is the same. The formulas, therefore, in the case of the Old Calendar are the following (*vid.* Art. 64):—

$$\epsilon = \left(\frac{11\,N - 3}{30}\right)_r \text{ (Arts. 64, 115).} \tag{1}$$

$$L = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w + 4}{7}\right)_r \text{ (Art. 92).} \tag{2}$$

$$\left.\begin{aligned} P &= \ 45 - \epsilon \\ \lambda &= \left(\frac{27 - \epsilon}{7}\right)_r \end{aligned}\right\} \text{ or, when } \epsilon > 23 \left\{\begin{aligned} &= \ 75 - \epsilon \\ &= \left(\frac{57 - \epsilon}{7}\right)_r. \end{aligned}\right. \tag{3}$$

$$L - \lambda. \tag{4}$$

$$\pi = P + L - \lambda. \tag{5}$$

143. Hence *for determining Easter Day of the Julian Calendar, in any year of the Old Style, we have the following Rule,* analogous to that for the Gregorian Calendar (Arts. 136, 138). The different steps are numbered, for the sake of comparison, to correspond with those of the Gregorian Rule.

1°. Find the Golden Number; $\left(\frac{x+1}{19}\right)_r$.

2°. Take 3 from 11 times 1°; divide by 30: the remainder is the Epact; or, if no remainder, 30 is the Epact; $\epsilon = \left(\frac{11 \times N - 3}{30}\right)_r$. (1)

3°, 4°, 5°, and 6°, involving σ, do not exist in the Old Calendar.

8°. The number of the given year; x.

9°. The quotient of the given year divided by 4, omitting fractions; $\left(\frac{x}{4}\right)_w$.

10°. Omit, as involving σ.

11°. Add 4 to the sum of 8° and 9°; divide by 7, and keep the remainder.

12°. Subtract 11° from 7, and compare the resulting number with the scale

A	B	C	D	E	F	G
1	2	3	4	5	6	7

This gives the Sunday Letter.

13°, 14°, 15°, same as in Gregorian Rule.

To the examples already given (Art. 64), I will add one or two more.

Ex. 1. Find Easter Day for A. D. 326 (Old Style).

Here we have $N = \text{IV.}$; $\epsilon = 11$; $L = 2 = \text{B}$.

$$\therefore P = 45 - 11 = 34.$$

$$\lambda = \left(\frac{27 - 11}{7}\right)_r = 2;$$

$$L - \lambda = 2 - 2 = 0.$$

$$\therefore \pi = 34 + 0 = \text{April } 3.$$

Ex. 2. Find Easter Day, A. D. 1001.

Here we have $N = \text{XIV.}$; $\epsilon = 1$; $L = 5 = \text{E}$.

$$\therefore P = 45 - 1 = 44$$

$$\lambda = \left(\frac{27 - 1}{7}\right)_r = 5$$

$$L - \lambda = 0$$

$$\therefore \pi = 44 + 0 = \text{April } 13.$$

Ex. 3. Find Easter Day for A. D. 1508.

Here we have $N = \text{VIII.}$; $\epsilon = 25$; $L = 1 = \text{A}$.

$\therefore P = 75 - 25 = 50$: change 25 to 26, which may always be done, and we get $P = 49$.

$$\lambda = \left(\frac{57 - 26}{7}\right)_r = 3$$

$$L - \lambda = 1 - 3 = -2 = 5$$

$$\therefore \pi = 49 + 5 = \text{23rd of April.}$$

(1). Delambre (Astr. Mod., i. 50) objects to the above expression for the Julian Epact, viz., $\left(\frac{11 \times N - 3}{30}\right)_r$; and accordingly gives another solution of the problem. But his objection is groundless; and the solution given above is far preferable to his.

His own formula (which he adopts from *L'art de Verifier les Dates*) is

$$\epsilon = \left(\frac{11\,x + \left(\frac{x-1}{19}\right)_w}{30}\right)_r.$$

He takes $M = \left(\frac{18 + 19\,N}{30}\right)_r$, and $\lambda = \left(\frac{M - 18}{7}\right)_r$.

Applying this to A.D. 326, we get $M = 34$, as above for P; $\lambda = 2$, as above, and find the same result.

Putting successively the different values of N in the formula $\left(\frac{18 + 19\,N}{30}\right)_r$, we get the following relations between N, M, and λ.

N	M	λ	N	M	λ
1	37	5	11	47	1
2	26	1	12	36	4
3	45	6	13	25	0
4	34	2	14	44	5
5	23	5	15	33	1
6	42	3	16	22	4
7	31	6	17	41	2
8	50	4	18	30	5
9	39	0	19	49	3
10	28	3			

M depends only on N; and all the numbers after the first thirty-seven are found by successive substitution of 11, and adding 30 when necessary.

λ depends on M, and is related to it by the formula

$$\lambda = \left(\frac{M-18}{7}\right)_r = \left(\frac{M-4}{7}\right)_r.$$

The period commenced in A. D. 608, the Golden Number of which was I.

144. Delambre (Astr. Mod., i., p. 22) has tabulated the 217 possible combinations of the 31 Epacts (25′ being reckoned as a separate one) with the 7 Sunday Letters; his table obviously corresponds to Clavius' (Art. 134).

Delambre's Table consists of *seven* principal columns, in the order of the seven Dominical Letters, beginning (like Clavius') with D, and in the order $\begin{matrix} \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{A} & \mathrm{B} & \mathrm{C} \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{matrix}$. Each of the seven columns is subdivided into four: the first, headed ϵ, giving the 30 Epacts—beginning with 23, which gives the earliest possible Easter, March 22; the second (λ) showing the values of λ (Art. 137) corresponding to the 31 Epacts, and the Sunday Letters of the respective columns; the third ($L - \lambda$) showing, similarly, the different values of $L - \lambda$ (Art. 137); and the fourth (π) giving the corresponding dates of Easter Day. Thus, for Epact 23, $L = \mathrm{D} = 4$, we get (Col. I.) $\lambda = 4$; $L - \lambda = 0$; π = March 22: for $L = \mathrm{E} = 5$, we get (Col. II.) for Epact 23, $\lambda = 4$; $L - \lambda = 1$; π = March 23; and so on. The following is

Delambre's Table for finding Easter.

Col. I. $L = D = 4$.

ϵ	λ	$L - \lambda$	π
23	4	0	22 M.
22	5	6	29
21	6	5	29
20	0	4	29
19	1	3	29
18	2	2	29
17	3	1	29
16	4	0	29
15	5	6	5 Apr.
14	6	5	5
13	0	4	5
12	1	3	5
11	2	2	5
10	3	1	5
9	4	0	5
8	5	6	12 Apr.
7	6	5	12
6	0	4	12
5	1	3	12
4	2	2	12
3	3	1	12
2	4	0	12
1	5	6	19 Apr.
0	6	5	19
29	0	4	19
28	1	3	19
27	2	2	19
26	3	1	19
25′	4	0	19
25	4	0	19
24	5	6	26 Apr.

26 April is beyond the Paschal limit: we must, therefore, go back to April 19, and 24 must be changed to 25. 25′ may be changed to 26 without any inconvenience.

Col. II. $L = E = 5$.

ϵ	λ	$L - \lambda$	π
23	4	1	23 M.
22	5	0	23
21	6	6	30
20	0	5	30
19	1	4	30
18	2	3	30
17	3	2	30
16	4	1	30
15	5	0	30
14	6	6	6 Apr.
13	0	5	6
12	1	4	6
11	2	3	6
10	3	2	6
9	4	1	6
8	5	0	6
7	6	6	13 Apr.
6	0	5	13
5	1	4	13
4	2	3	13
3	3	2	13
2	4	1	13
1	5	0	13
0	6	6	20 Apr.
29	0	5	20
28	1	4	20
27	2	3	20
26	3	2	20
25′	4	1	20
25	4	1	20
24	5	0	20

We may here change 24 into 25, and 25′ into 26, without any inconvenience.

Col. III. $L = F = 6$.

ϵ	λ	$L - \lambda$	π
23	4	2	24 M.
22	5	1	24
21	6	0	24
20	0	6	31 M.
19	1	5	31
18	2	4	31
17	3	3	31
16	4	2	31
15	5	1	31
14	6	0	31
13	0	6	7 Apr.
12	1	5	7
11	2	4	7
10	3	3	7
9	4	2	7
8	5	1	7
7	6	0	7
6	0	6	14 Apr.
5	1	5	14
4	2	4	14
3	3	3	14
2	4	2	14
1	5	1	14
0	6	0	14
29	0	6	21 Apr.
28	1	5	21
27	2	4	21
26	3	3	21
25′	4	2	21
25	4	2	21
24	5	1	21

Same remark as the last.

Col. IV. $L = G = 7$.

ϵ	λ	$L - \lambda$	π
23	4	3	25 M.
22	5	2	25
21	6	1	25
20	0	0	25
19	1	6	1 Apr.
18	2	5	1
17	3	4	1
16	4	3	1
15	5	2	1
14	6	1	1
13	0	0	1
12	1	6	8 Apr.
11	2	5	8
10	3	4	8
9	4	3	8
8	5	2	8
7	6	1	8
6	0	0	8
5	1	6	15 Apr.
4	2	5	15
3	3	4	15
2	4	3	15
1	5	2	15
0	6	1	15
29	0	0	15
28	1	6	22 Apr.
27	2	5	22
26	3	4	22
25′	4	3	22
25	4	3	22
24	5	2	22

Same remark.

Col. V. $L = A = 1$.

ϵ	λ	$L - \lambda$	π
23	4	4	26 M.
22	5	3	26
21	6	2	26
20	0	1	26
19	1	0	26
18	2	6	2 Apr.
17	3	5	2
16	4	4	2
15	5	3	2
14	6	2	2
13	0	1	2
12	1	0	2
11	2	6	9 Apr.
10	3	5	9
9	4	4	9
8	5	3	9
7	6	2	9
6	0	1	9
5	1	0	9
4	2	6	16 Apr.
3	3	5	16
2	4	4	16
1	5	3	16
0	6	2	16
29	0	1	16
28	1	0	16
27	2	6	23 Apr.
26	3	5	23
25′	4	4	23
25	4	4	23
24	5	3	23

Same remark.

Col. VI. $L = B = 2$.

ϵ	λ	$L - \lambda$	π
23	4	5	27 M.
22	5	4	27
21	6	3	27
20	0	2	27
19	1	1	27
18	2	0	27
17	3	6	3 Apr.
16	4	5	3
15	5	4	3
14	6	3	3
13	0	2	3
12	1	1	3
11	2	0	3
10	3	6	10 Apr.
9	4	5	10
8	5	4	10
7	6	3	10
6	0	2	10
5	1	1	10
4	2	0	10
3	3	6	17 Apr.
2	4	5	17
1	5	4	17
0	6	3	17
29	0	2	17
28	1	1	17
27	2	0	17
26	3	6	24 Apr.
25′	4	5	24
25	4	5	24
24	5	4	24

Same remark.

Col. VII. $L = C = 3$.

ϵ	λ	$L - \lambda$	π
23	4	6	28 M.
22	5	5	28
21	6	4	28
20	0	3	28
19	1	2	28
18	2	1	28
17	3	0	28
16	4	6	4 Apr.
15	5	5	4
14	6	4	4
13	0	3	4
12	1	2	4
11	2	1	4
10	3	0	4
9	4	6	11 Apr.
8	5	5	11
7	6	4	11
6	0	3	11
5	1	2	11
4	2	1	11
3	3	0	11
2	4	6	18 Apr.
1	5	5	18
0	6	4	18
29	0	3	18
28	1	2	18
27	2	1	18
26	3	0	18
25′	4	6	25 Apr.
25	4	6	25
24	5	5	25

25′ *must* be changed to 26, and 24 *may* be changed to 25.

145. The inspection of this Table shows that λ *increases* regularly from 0 to 6; and the $\epsilon + \lambda$ is constant during each cycle of λ: *e.g.*, Col. I., $\lambda = 0$ occurs with $\epsilon = 20$; and, ϵ decreasing as fast as λ increases, their sum is constant (= 20 for this series). On the other hand, $L - \lambda$ *decreases* regularly from 6 to 0.

We see in Col. I. that with Epact 24 and Sunday Letter D, Easter Day would fall on April 26 (Art. 141); which is beyond the Paschal limit. We *must* therefore change 24 into 25, which gives April 19 for Easter Day.

In the same Column, I., we see that 25′ may be changed into 26 without altering the result.

In Columns II., III., IV., V., VI., 25 *may* be used for 24, and 26 for 25′.

In Col. VII., Sunday Letter C, 24 *may* be changed to 25; but 25′ must be changed to 26: otherwise, as 25′ and 24 are in the *same* cycle, two Easters would fall on the *same* day (April 25). (Art. 141).

We see also (Col. I.), as before, that only one Epact (23) will give Easter Day on March 22: and this only when L is D.

In Col. VII. we see that two Epacts, 25 and 24, will give the other extreme Easter, April 25.

When the Epact and Sunday Letter are given, the above Table gives at once Easter Day.

Ex. 1. Find Easter Day for 1876.

The Epact is 4, and Sunday Letter A.

Then we see in Col. V. that opposite Epact 4 we find Easter Day April 16, as it ought, for the Table gives $\lambda = 2$ and $L - \lambda = 6$.

$$\therefore P = 45 - \epsilon = 41,$$
$$\text{and } \pi = P + L - \lambda = 41 + 6 = 47 = \text{April 16.}$$

Ex. 2. Given Epact = 24, and Sunday Letter E.

Looking into Col. II. (E) we find opposite ϵ, 24, April 20, or Easter Day; which agrees with the general rule: for $\lambda = 4$ and $L - \lambda = 0$.

$$P = 75 - 24 = 51,$$
$$\therefore \pi = P + L - \lambda = 51 + 0 = 51 = \text{April 20.}$$

146. This Table can also be used for finding the Old Style Easter, when the Julian Epact (ϵ') and Sunday Letter (L') are given; because, as we have seen already (Arts. 64, 137), the very same formula applies to both Easters, viz.:

$$\pi = P + L - \lambda\text{; Gregorian Easter.}$$
$$\pi' = P' + L' - \lambda\text{; Julian Easter.}$$

Ex. Find by the Table the Julian or O. S. Easter for the year A. D. 1880.

Here $$\text{G. N.} = \left(\frac{x+1}{19}\right)_r = \text{XIX}.$$

$$\epsilon' = \left(\frac{11 \times N - 3}{30}\right)_r = 26.$$

$$L' = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w + 4}{7}\right)_r = 5 = \text{E}.$$

Look in Col. II., under Epact 26, and we find Easter Day April 20.

147. We have seen (Art. 70) the long intervals which take place between the occurrence of Easter Day at the extreme limits of March 22 and April 25, in the Old Calendar; and the law which these intervals follow has also been investigated (Art. 71). Let us now inquire how these intervals are modified in the Gregorian Calendar.

The following Table shows the years in which Easter Day occurs on March 22 and April 25, respectively, from the reformation of the Calendar in 1582 to A. D. 5000.

Easter on March 22.	Intervals.	Easter on April 25.	Intervals.	Easter on April 25.	Intervals.
A. D.		A. D.		A. D.	
1598		1666		3097	
	95 = 19 × 5		68 = 19 × 3 + 11		57 = 19 × 3
1693		1734		3154	
	68 = 19 × 3 + 11		152 = 19 × 8		95 = 19 × 5
1761		1886		3249	
	57 = 19 × 3		57 = 19 × 3		57 = 19 × 3
1818		1943		3306	
	467 = 19 × 24 + 11		95 = 19 × 5		163 = 19 × 8 + 11
2285		2038		3469	
	68 = 19 × 3 + 11		152 = 19 × 8		68 = 19 × 3 + 11
2353		2190		3537	
	84 = 19 × 5 − 11		68 = 19 × 3 + 11		84 = 19 × 5 − 11
2437		2258		3621	
	68 = 19 × 3 + 11		68 = 19 × 3 + 11		163 = 19 × 8 + 11
2505		2326		3784	
	467 = 19 × 24 + 11		84 = 19 × 5 − 11		209 = 19 × 11
2972		2410		3993	
	57 = 19 × 3		163 = 19 × 8 + 11		95 = 19 × 5
3029		2573		4088	
	372 = 19 × 19 + 11		57 = 19 × 3		68 = 19 × 3 + 11
3401		2630		4156	
	95 = 19 × 5		152 = 19 × 8		68 = 19 × 3 + 11
3496		2782		4224	
	68 = 19 × 3 + 11		95 = 19 × 5		152 = 19 × 8
3564		2877		4376	
	84 = 19 × 5 − 11		68 = 19 × 3 + 11		152 = 19 × 8
3648		2945		4528	
	68 = 19 × 3 + 11		57 = 19 × 3		152 = 19 × 8
3716		3002		4680	
	592 = 19 × 31 + 3		95 = 19 × 5		68 = 19 × 3 + 11
4308		3097		4748	
					152 = 19 × 8
				4900	

Clavius (p. 514) gives for 3860, March 22; but this is clearly a misprint for April 22. In the first place, in the Extended Table of Epacts, A. D. 3860, the Epact 23 (an essential condition) does not appear. Again, A. D. 3860 has Epact 24, Sunday Letter G (D being the condition for March 22). And, lastly, Epact 24, G, are the conditions for April 22 (Table, Art. 144). Delambre follows this mistake or misprint, and observes that the interval between 3860 and 4308 is $448 = 19 \times 23 + 11$. Omitting, therefore, 3860, the interval between 3716 and $4308 = 592 = 19 \times 31 + 3$. The next occurrence, after 4308, of March 22 is 5299, between which and 4308 the interval is $991 = 19 \times 51 + 3$ (1).

From the above Table the following results follow :—

1°. From the reformation of the Calendar (1582) down to A. D. 5000, Easter falls on March 22 (earliest day) only sixteen times.

2°. During the same time, Easter falls on April 25 (latest day) thirty-two times, or twice as often. The reason of this is that while only *one* Epact (23) will give March 22, *two* Epacts (24 and 25) will give April 25.

The reason why in the Old Julian Calendar the extreme Easters occur the same number of times is that in this case only *one* Golden Number, VIII., with its corresponding Epact, 25, will give April 25; whereas in the Gregorian Calendar 24 also will give it.

3°. The intervals between the consecutive occurrences are, in both cases, of the form $19 \times n$, or $19 \times n \pm 11$, with exception of the last interval in the first series, which is $19 \times 51 + 3$.

4°. In the former case n is 3, 5, 19, 24, 31; in the latter case, 3, 5, 8, 11. The *shortest* interval in each case is 57 years $= 19 \times 3$.

5°. The longest intervals, 372, 467, 592, 991, are peculiar to the limit of March 22.

Delambre observes that in neither series can any regular *law* of succession be traced, such as we have found to prevail in the Old Calendar (Art. 71).

I have calculated the cases of Easter falling on March 22, from A. D. 5000 to 8500, and find as follows:—

A. D.	INTERVALS.
4308	—
5299	991 = 19.52 + 3
5671	372 = 19.19 + 11
6043	372 = 19.19 + 11
6195	152 = 19.8
6263	68 = 19.3 + 11
6415	152 = 19.8
6635	220 = 19.11 + 11
6703	68 = 19.3 + 11
6798	95 = 19.5
6882	84 = 19.5 − 11
6950	68 = 19.3 + 11
7322	372 = 19.19 + 11
7474	152 = 19.8
7542	68 = 19.3 + 11
7637	95 = 19.5
7789	152 = 19.8
7914	125 = 19.6 + 11
8161	247 = 19.13
8533	372 = 19.19 + 11

There is only one case of the law $19n$ or $19n \pm 11$ being interrupted, viz., the interval 991. In century 67 there are two cases of earliest Easter.

148. *To find whether in a given century there will be any and what cases of the occurrence of Easter on either of the extreme limits, March 22, April 25.*

I. *March* 22.—The necessary and sufficient conditions for this are Epact 23, and Sunday Letter D.

First; look for the given *century* in the *Index* Column of the *Expanded Table of Epacts* (Art. 123), and see whether 23 occurs in the line of Epacts belonging to that century; and if so, note the corresponding Golden Number at the head of the column.

Secondly; look in the Table of *Golden Numbers* (Art. 58), under "Centurial Years," for the given *century*, and in the corresponding line of Golden Numbers find the above Golden Number; and in the same column of the uppermost smaller Table, note the different *years* of the given century, which have that Golden Number.

Thirdly; in the Table of Gregorian *Sunday Letters* (Art. 95) look for these years just found, and observe which of them, if any, have the Sunday Letter D. These years will have Easter day on March 22.

Ex. 1. Find whether Easter will fall on March 22 in the century 1600–1699.

First; we see in the Expanded Table of Epacts, that the line of Epacts opposite Index 16 does contain 23; and that the Golden Number at the top of the column is III.

Secondly; in the Table of Golden Numbers, "Centurial Years" we look for 16, and in the same row for Golden Number III.; and at the top of the column we find 17, 36, 55, 74, 93; showing that 1617, '36, '55, '74 and '93 have the Golden Number III.

Thirdly; in the Table of Sunday Letters we look, under "Hundreds of Years" for 16, and, in the "Units of Tens" for 17, 36, 55, 74, 93; and we find that the last year alone (1693) has its Sunday Letter D. Hence, there was in the given century (1600–1699) only *one* year (1693) in which Easter fell on March 22. Proceeding in the same way, we find that there was *one* in 1598, 1761, 1818; and *none* in the centuries 19, 20, and 21, because under those Indices in the Expanded Table of Epacts, the Epact 23 is not found. We also find that in *one* case only, two Easters, March 22, fall in the same century, viz., 3401, 3496 (*vid.* below).

II. Easter on April 25. The necessary and sufficient conditions are, Epacts 24 or 25, and Sunday Letter C. Now, looking at the Expanded Table of Epacts, we see that in every one of the thirty lines, either (not both) 24 or 25 occurs; so that this condition it always satisfied. And, again, with regard to the other condition (L = C), we find on examination (as the above Table in this Article shows) that there are only four centuries in which this condition also is *not* satisfied, viz., 38, 44, 48, 50. We find, further, that

2 F

there is only *one* century (as in the case of March 22) in which *two* Easters fall on April 25, viz., 30, the years being 3002 and 3097.

To find the year or *years* in any century in which Easter will fall on April 25, we proceed in an exactly similar way to that followed in the case of March 22.

Ex. Find whether any and what year in the century 1800–99 will have Easter Day on April 25.

We find opposite Index 18 (Expanded Table) Epact 25, with Golden Number VI. In Table of Golden Numbers we find that 1800, VI. gives the years 10, 29, 48, 67, 86, in which the Golden Number for that century will be VI. Then, in Table of Sunday Letters we see that the only one of these six years which has Sunday Letter C is 1886, which is the required year.

Ex. Find Easter Day A. D. 3401.

Here we have

$$\begin{aligned} \epsilon &= 23,\ L = \mathrm{D} = 4 \\ \therefore P &= 45 - 23 = 22 \\ \lambda &= 27 - 23 = 4 \\ \therefore L - \lambda &= 0 \\ \therefore \pi - P + L - \lambda &= 22 + 0 = \text{March } 22. \end{aligned}$$

Find Easter Day A. D. 3496.

Here, also,

$$\begin{aligned} \epsilon &= 23,\ L = \mathrm{D} = 4 \\ \therefore P &= 45 - 23 = 22 \\ \lambda &= 27 - 23 = 4 \\ L - \lambda &= 0 \\ \therefore \pi = P + L - \lambda &= 22 + 0 = \text{March } 22. \end{aligned}$$

Find Easter Day A. D. 3097.

Both of these years have

$$\begin{aligned} \epsilon &= 25 \text{ and } L = \mathrm{C} = 3 \\ P &= 75 - 25 = 50 \\ \lambda &= \left(\frac{57 - 25}{7}\right)_r = 4 \\ \therefore L - \lambda &= 3 - 4 = -1 = 6 \\ \therefore P &= 50 + 6 = 56 \text{ of March} = \text{April } 25. \end{aligned}$$

149. I alluded (Art. 135) to Gauss' solution of the Easter problem, given in Zach's *Monatliche Correspondenz*, 1800. The rule which he gives apparently dispenses with the Golden Numbers and Epacts, and Sunday Letters, and requires the year A. D. only to be given. But the elements just mentioned are implicitly involved in the quantities which he denotes by M and N. His Rule is this. Let x be the number of the given year A. D.

1°. Divide x by 19, and let a be the remainder; $\left(\frac{x}{19}\right)_r = a$

2°. " 4, " b " " ; $\left(\frac{x}{4}\right)_r = b$

3°. " 7, " c " " ; $\left(\frac{x}{7}\right)_r = c$

4°. To 18 times a add M; divide the sum by 30, and let the remainder be d;

$$\left(\frac{19 \times \left(\frac{x}{19}\right)_r + M}{30}\right)_r = d;$$

5°. To twice b add 4 times c, and 6 times d: add N to this sum, and divide by 7, and let the remainder be e.

$$\left(\frac{2\left(\frac{x}{4}\right)_r + 4\left(\frac{x}{7}\right)_r + 6\left(\frac{19\left(\frac{x}{19}\right)_r + M}{30}\right)_r + N}{7}\right)_r = e.$$

6°. In the *Julian* Calendar make $M = 15$, and $N = 6$.

In the *Gregorian* Calendar from

1582–1699	make M =	22,	N =	3
1700–1729	"	23,	"	3
1800–1899	"	23,	"	4
1900–1999	"	24,	"	5
2000–2099	"	24,	"	5
2100–2199	"	24,	"	6
2200–2299	"	25,	"	0
2300–2399	"	26,	"	1
2400–2499	"	25,	"	1

to find Easter Day from the above.

It will be $(22 + d + e)$, of *March*; that is, $(d + e)$ days after March 22, or $(d + e - 9)$ of April.

This Rule holds good *without exception* for the Julian Calendar; and in the *Gregorian* it admits of but two exceptions.

1°. If the calculation gives April 26, substitute the preceding Sunday, 19th,

2°. If the calculation gives April 25, substitute the preceding Sunday, 18th, when the Golden Number is greater than eleven.

Ex. 1. Required Easter Day for A. D. 1812.

Here we have

$$a = 7\,;\ b = 0\,;\ c = 6\,;\ M = 23\,;\ N = 4\,;\ d = 6\,;\ e = 1.$$

Hence, $22 + 6 + 1 = 29$th March.

Ex. 2. Find Easter Day for 1876.

6. We have, as before, $M = 23$, $N = 4$.

1. $\left(\frac{1876}{19}\right)_r = 14$, a.

2. $\left(\frac{1876}{4}\right)_r = 0$, b.

3. $\left(\frac{1876}{7}\right)_r = 0$, c.

4. $\left(\frac{23 + 19 \times 14}{30}\right)_r = 19$, d.

5. $\left(\frac{4 + 0 + 0 + 6 \times 19}{7}\right)_r = 6$, e.

Therefore, Easter Sunday is March $22 + 19 + 6 =$ March 47, or April 16; or, $19 + 6 - 9$ of April, = April 16.

Ex. 3. Find Easter Day for A. D. 2133.

Here, $M = 24$; $N = 6$; $L = D = 4$.

$$\left(\frac{2133}{19}\right)_r = 5, \quad a.$$

$$\left(\frac{2133}{4}\right)_r = 1, \quad b.$$

$$\left(\frac{2133}{7}\right)_r = 5, \quad c.$$

$$\left(\frac{24 + 95}{30}\right)_r = 29, \quad d.$$

$$\left(\frac{2 + 20 + 174 + 6}{7}\right)_r = 6, \quad e.$$

Easter Day is $22 + 29 + 6 = 57$ of March = 26 April, which must, as above stated, be changed into April 19.

Ex. Find Easter Day, A. D. 1954.

Here, $M = 24$; $N = 5$.

$$\left(\frac{1954}{19}\right)_r = 16, \quad a.$$

$$\left(\frac{1954}{4}\right)_r = 2, \quad b.$$

$$\left(\frac{1954}{7}\right)_r = 1, \quad c.$$

$$\left(\frac{24 + 304}{30}\right)_r = 28, \quad d.$$

$$\left(\frac{4 + 4 + 168 + 5}{7}\right)_r = 6, \quad e.$$

Therefore, Easter Day $= 22 + 28 + 6 = 56$ of March = 25th of April. But as the Golden Number (XVII.) is > 11. The preceding Sunday must be taken, as above noted, viz., April 19.(1)

(1). Cicoolini deduces from his formula for the Epact (*vid.* Note to Art. 136), viz.,

$$\epsilon = \left(\frac{11\,N - \left(\frac{3\sigma - 5}{4}\right)_w + \left(\frac{8\sigma - 112}{30}\right)_w}{30}\right)_r$$

(σ denoting the number of the century) – a Rule for calculating Easter, very similar in form to that of Gauss: thus, put $\rho = 30 - \epsilon$; $d = \left(\frac{23 + \rho}{30}\right)_r$; $e = \left(\frac{3 + L + 6.\,d}{7}\right)_r$ where L = number of Sunday Letter of the year.

Easter Day will be $22 + d + e$ of March.
or $d + e - 9$ of April.

It is subject to the same two exceptions as Gauss' Rule. Delambre, Astr. Mod. 1, 52.

Ex. 1. Find Easter Day A. D. 2285.

N = VI., and $\sigma = 22$, $L = D \quad 4$.

Hence, $\epsilon = 23$; $\rho = 7$; $d = 0$; $e = 0$,

$\therefore$ Easter Day $= 22 + 0 + 0 =$ March 22.

Ex. 2. Find Easter Day A. D. 1943.

$$N = \text{VI}\,;\ L = C = 3\,;\ \therefore\ \epsilon = 24,\ \rho = 6\,;\ d = 29\,;\ e = 5.$$

Hence, Easter Day is $22 + 29 + 5 = 56$ of March, 25th of April,

or $29 + 5 - 9 = 25$th of April.

150. A proof of Gauss' Rule for the two centuries, 1700–1899, is given by himself in *Zach's Monatliche Correspondenz*, Vol. ii., p. 221, *sqq.*, Aug. 1800. His Rule concisely stated is as follows:—

Let A be the number of the year:—

1. Divide A by 19, and call the remainder a.
2. " 4, " " b.
3. " 7, " " c.
4. Divide the number $M + 19a$ by 30, and call the remainder d.
5. Divide the number $(2b + 4c + 6d + N)$ by 7, and call the remainder e.

Easter Day is $22 + d + e$ of March, or $d + e - 9$ of April. M and N are thus found:

Divide n by 100; and call the integer quotient k.
" k by 3; " " " p.
" k by 4; " " " q.

Then,
$$M = \left(\frac{15 + k - p - q}{30}\right)$$

$$N = \left(\frac{4 + k - q}{7}\right)_r.$$

Delambre (Astr. Theor. et Prat. iii., 712) incorrectly stated that Gauss had not given the law of the formation of the quantities M and N, and had merely stated their values as far as A. D. 2499; but he acknowledges his mistake in the *Additions to the Connaissance des Tems, for 1817, p. 315.*

The Rule as above given is subject to two exceptions:—

1°. If the calculation give April 26 for Easter Day, we must take the preceding Sunday in April 19.

2°. If the calculation give April 25 for Easter Day, and at the same time a be greater than 10, $d = 28$, and $e = 6$, we must take April 18 instead.

This latter exception is due to Clavius' artificial mode of contriving to avoid counting the same day of the month as the day of the Paschal Full Moon in two different years of the same Cycle of 19 years. The former exception is a real and necessary exception.

The following proof is substantially the same as Gauss' original proof, but is applicable to any century.

I. The Easter Full Moon (14th day), for the year whose Golden Number is I., falls in the seventeenth century on April 12; in the eighteenth and nineteenth centuries on April 13; and in general, if $\odot$ be the Solar, ☽ the Lunar, Equation for any century, the Easter Full Moon will fall on March $(21 + D)$, when $D = \left(\frac{22 + \odot - \text{☽}}{30}\right)_r$, in the years whose Golden Number is I.

For every successive year throughout the 19-year Cycle, it will fall either 11 days earlier, or 19 days later, than in the foregoing year, according as the Easter of the previous year falls in April or March (see Art. 113). Hence, it follows that the Easter Full Moon (14th day) never falls before March 21, or later than April 19.

From what has just been said, it follows that for any year $a + 1$ of the 19 year Cycle, the Easter Full Moon will fall on March $21 + D$, where $D = \left\{\frac{22 + \odot - \text{☽} - 11\,x + 19\,y}{30}\right\}_r$

and $x + y = a$. Substituting for x, we have $D = \left\{\frac{19\,a + 22 + \odot - \text{☽}}{30}\right\}_r$.

Now, by previous results $\odot = \sigma - 16 - \left(\frac{\sigma - 16}{4}\right)_w$, where $\sigma = k$ of this Article.

$$\text{☽} = \left(\frac{\sigma - 15 - \left(\frac{\sigma - 17}{25}\right)_w}{3}\right)_w$$

$$\therefore \odot - \text{☽} = \sigma - \left(\frac{\sigma}{4}\right)_w - 7 - \left(\frac{\sigma + 1 - \left(\frac{\sigma + 8}{25}\right)_w}{3}\right)_w$$

$$= k - q - 7 - \left(\frac{k + 1 - \left(\frac{k + 8}{25}\right)_w}{3}\right)_w;$$

and
$$\therefore D = \left\{\frac{15 + 19\,a + k - q - \left(\frac{k + 1 - \left(\frac{k + 8}{25}\right)_w}{3}\right)_w}{30}\right\}_r.$$

Comparing this with Gauss' d, we see that he omits the consideration of $1 - \left(\frac{k+8}{25}\right)_w$,

and writes $\left(\frac{k}{3}\right)_w = p$, instead of $\left(\frac{k + 1 - \left(\frac{k+8}{25}\right)_w}{3}\right)_w$.

Doubtless, the reason why Gauss omitted the consideration of this quantity, thus regarding the Lunar Equation as a correction of one day *every* 300 years, is because the difference between an allowance of one day in 300 years, and eight days in 2500 years, only amounts *on an average* to one day in 7500 years: whereas the error of the Gregorian Calendar is about one day in 3600 years. But as the Calendar is constituted, we shall require the use of the true value of p in the year 4200, A. D., by which time the Gregorian Calendar will *not* yet be one day in error.

In either case, whether p be considered, with Gauss, as equal to

$\left(\frac{k}{3}\right)_w$, or as $\left(\frac{k + 1 - \left(\frac{k+8}{25}\right)_w}{3}\right)_w$, we may write $d = \left(\frac{15 + 19\,a + k - q - p}{30}\right)_r$,

and date of Easter Full Moon (14th day) is March $(21 + d)$.

II. Easter itself falls on the *first* Sunday after the 14th day of the Moon; *i.e.*, at least one, or at most seven days after. Supposing, then, that Easter falls on March $(22 + d + E)$, E will be between 0 and 6 inclusive. It is defined by the condition that it must be a *Sunday*. This condition may be arithmetically stated as follows:

The interval between March $(22 + d + E)$ of the given year, and any other definite Sunday, must be a multiple of 7. We must then take some particular Sunday. Gauss takes Sunday, March 21, 1700. Let the number of the given year be A; i the number of Leap years, between 1700 (exclusive) and A (inclusive, if it be a Leap year): accordingly, i is the number of days intercalated in the interval between March 21, 1700, and Easter Day of year A.

Hence, the total number of days from March 21, 1700, to March $(22 + d + E)$ of the given year, A is

$$= 1 + d + E + i + 365\,(A - 1700):$$

but
$$i = \left(\frac{A - 1700}{5}\right)_w - \sigma - 16 + \left(\frac{\sigma - 16}{4}\right)_w + 1,$$

but
$$\left(\frac{A}{4}\right)_w = \frac{A - \left(\frac{A}{4}\right)_r}{4} = \frac{A - b}{4}$$

$$i = \tfrac{1}{4}\,(A - b - 1700) + 13 - k + q.$$

Thus we must have, adding $\frac{7}{4}\,(A - b - 1700)$ (a mult. of 7)

$$14 + d + E + 367\,(A - 1700) - 2\,b - k + q = \text{mult. } 7,$$

or omitting $364\,(A - 1700)$

$$3\,A + d + E - 2\,b - k + q - 5100 = \text{mult. } (7),$$

or subtracting $3\,A - 3\,c$ which is divis. by 7, we must have

$$3\,c + d + E - 2\,b - k + q - 4 = \text{mult. } (7),$$

subtract this from $7\,c + 7\,d$, and we have finally $4 + 4\,c + 6\,d + 2\,b + k - q - E$ divisible by 7, and hence

$$E = \left(\frac{2b + 4c + 6d + 4 + k - q}{7}\right)_r$$

$$= \left(\frac{2\,b + 4\,c + 6\,d + N}{7}\right)_r = e.$$

Thus Gauss' Rule is completely proved.[1]

(1). A New York correspondent sent without proof the following rule to "NATURE," April 20, 1876, p. 485. It has the advantage of being subject to no exceptions, a special provision having been made to guard against the two ordinary exceptional cases.

The Rule is as follows:—

Divide	By	And call the Quotient	Remainder.
The given year n . . .	19	—	a
"	100	b	c
b	4	d	e
$b + 8$	25	f	—
$b - f + 1$	3	g	—
$19\,a + b - d - g + 15$ . .	30	—	h
c	4	i	k
$32 + 2\,e + 2\,i - h - k$ . .	7	—	l
$a + 11\,h + 22\,l$	451	m	—
$h + l - 7\,m + 114$ . . .	31	n	o

n is the number of the month of the year, and $o + i$ is the number of the day of the month on which Easter falls.

I will now show how this Rule may be deduced from Delambre's Analytical Solution.

Delambre's analysis may be summarised thus:—

$$N = \left(\frac{x}{19}\right)_r + 1$$

$$L = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w + 1 - \overline{\sigma - 16} + \left(\frac{\sigma - 16}{4}\right)_w}{7}\right)_w$$

$$\varepsilon = \left(\frac{11\,N - 10}{30}\right)_r - \overline{\sigma - 16} + \left(\frac{\sigma - 16}{4}\right)_w + \left(\frac{\sigma - 15 - a}{3}\right)_w$$

where $$a = \left(\frac{\sigma - 17}{25}\right)_w$$

$$P = 22 + \left(\frac{53 - \varepsilon}{30}\right)_r$$

$$\lambda = \left(\frac{4 + \left(\frac{53 - \varepsilon}{30}\right)_r}{7}\right)_w$$

$$\pi = P + \left(\frac{7 + L - \lambda}{7}\right)_r \text{ day of March.}$$

Now $$N = \left(\frac{x}{19}\right)_r + 1 = a + 1. \qquad (1)$$

$$x = 100b + c \qquad \therefore \sigma = b. \qquad (2)$$

$$b = 4d + e \qquad \therefore \left(\frac{\sigma}{4}\right)_w = d. \qquad (3)$$

$$c = 4i + k \qquad \therefore \left(\frac{c}{4}\right)_w = i. \qquad (4)$$

$$b + 8 = 25f + \ldots . \qquad \therefore \left(\frac{\sigma + 8}{25}\right)_w = f. \qquad (5)$$

$$b + 1 - f = 3g + \ldots \therefore \left(\frac{\sigma + 1 - \left(\frac{\sigma + 8}{25}\right)_w}{3}\right)_w = g = \left(\frac{\sigma - \left(\frac{\sigma - 17}{25}\right)_w}{8}\right)_w. \qquad (6)$$

Now $$\varepsilon = \left(\frac{11\,N - 10}{30}\right)_r - \sigma + \left(\frac{\sigma}{4}\right)_w + 12 + \left(\frac{\sigma - \left(\frac{\sigma - 17}{25}\right)_w}{3}\right)_w - 5$$

$$= \left(\frac{11\,a - 1}{30}\right)_r - b + d + 7 + g$$

$$\therefore \left(\frac{53 - \epsilon}{30}\right)_r = \left(\frac{46 + b - d - g - \left(\frac{11\,a + 1}{30}\right)_r}{30}\right)_r$$

but
$$\left(\frac{11\,a + 1}{30}\right)_r + \left(\frac{19\,a - 1}{30}\right)_r = 0 \text{ or } 30;$$

$$\therefore \left(\frac{53 - \epsilon}{30}\right)_r = \left(\frac{19\,a + b - d - g + 15}{30}\right)_r = h. \tag{7}$$

Hence
$$\lambda = \left(\frac{4 + h}{7}\right)_r$$

$$L = 7 - \left(\frac{x + \left(\frac{x}{4}\right)_w + 13 - \sigma + \left(\frac{\sigma}{4}\right)_w}{7}\right)_r$$

$$= 7 - \left(\frac{125\,b + c + i + 13 - b + d}{7}\right)_r = 7 - \left(\frac{5\,b + c + d + i + 13}{7}\right)_r$$

$$\left(\frac{7 + L - \lambda}{7}\right)_r = \left(\frac{14 - \left(\frac{5\,b + c + d + i + 13}{7}\right)_r - \left(\frac{4 + h}{7}\right)_r}{7}\right)_r$$

$$= 7 - \left(\frac{5\,b + c + d + i + h + 17}{7}\right)_r \quad \text{but } b = 4\,d + e,\ c = 4\,i + k$$

$$= 7 - \left(\frac{5\,i + 5\,e + h + k + 17}{7}\right)_r$$

but
$$\left(\frac{5\,i + 5\,e + h + k + 17}{7}\right)_r + \left(\frac{32 + 2\,i + 2\,e - h - k}{7}\right)_r = 0 \text{ or } 7.$$

[I insert 32, as i and e may $= 0$, and $h + k$ may $= 32$].

Hence, substituting,

$$\left(\frac{7 + L - \lambda}{7}\right)_r = \left(\frac{32 + 2\,i + 2\,e - h - k}{7}\right)_r = l. \tag{8}$$

Hence, subject to Delambre's two exceptions

Easter $= 22 + h + l$ day of March.

$$\left.\begin{array}{l} \text{But } 1^\circ \text{ when } \epsilon = 24, \text{ and } L - \lambda = 6 \\ \quad 2^\circ \text{ when } \epsilon = 25, \text{ and } L - \lambda = 6 \text{ and G. N.} > 10 \end{array}\right\} \text{we must subtract 7.}$$

Hence, if m be such a quantity that

$$\left.\begin{array}{l} 1^\circ \text{ when } h = 29 \\ \qquad\quad l = 6 \\ \qquad\quad a = \text{anything} \\ 2^\circ \text{ when } h = 28 \\ \qquad\quad l = 6 \\ \qquad\quad a > 10 \end{array}\right\} m = 1.$$

3° In all other cases $m = 0$, we shall have for day of March

$$22 + h + l - 7\,m\,;$$

or since when $22 + h + l - 7\,m \leqq 31$, the month of the year is 3,

and when $22 + h + l - 7\,m > 31$, the month of the year is 4,

month of year is given by $\left(\dfrac{92 + 22 + h + l - 7\,m}{31}\right)_w = n,$ (9)

and day of month by $\left(\dfrac{92 + 22 + h + l - 7\,n}{31}\right)_r + 1 = o + i.$ (10)

It remains to find the quantity m.

Let us assume it to be of the form

$$\left(\frac{a + x.h + y.l}{29\,x + 6\,y}\right)_w.$$

I take this form because it is obvious that this satisfies condition 1° completely.

Taking condition 2°, we see that

$$11 + 28\,x + 6\,y \geqq 29\,x + 6\,y,$$

$$i.e.,\ x \leqq 11. \qquad \text{(A)}$$

Taking condition 3° and supposing as an extreme case that $h = 29$, $l = 5$, $a = 18$, then we must have

$$29\,x + 6\,y > 18 + 29\,x + 5\,y \;\therefore\; y > 18 \qquad \text{(B)}$$

Again, suppose that $h = 28$, $l = 6$, $a = 10$, we must have

$$29\,x + 6\,y > 10 + 28\,x + 6\,y, \;\therefore\; x > 10. \qquad \text{(C)}$$

From (A) and (C) we see that x must $= 11$,

y = any number greater than 18.

$= 19$, say.

Then $$m = \frac{a + 11\,h + 19\,l}{433}.$$

There is no reason why the value 22 should be given to the coefficient of l rather than any other value greater than 18.

151. The Gregorian reformation of the old Church Calendar consisted, as we have seen, of two very distinct parts, which may be called the *retrospective* and the *prospective*. Each of these parts again included two elements, also perfectly distinct from each other—the Solar and the Lunar. The former was concerned with the true Equinox; in other words, the problem was to restore the Calendar Equinox to the same day as that on which the true Equinox fell, and by suitable adjustment to keep the two as nearly as possible together for the future. The former object was attained by omitting the *ten* days, the latter by the *secular Solar Equation.* The omission of the ten days led to a displacement of the old Dominican Letters, and the Solar Equation requires a corresponding re-adjustment of them from age to age. The other element referred to is the 19 year Lunar Cycle, employed for the determination of Easter. The correction of the accumulated error of this Cycle, and the prospective adjustment of the Calendar Moon by the Lunar Equation involves intricate and difficult calculations in the determination of *Easter*, which render the subject almost unintelligible to the great majority of people. Hence it is much to be regretted that Pope Gregory did not see fit to adopt the most important of all reformations of the old Calendar, viz., making *Easter an immoveable Festival*, by fixing it to some particular Sunday in the year ([1]), say the last in March, or the first in April. At present very few, even among educated men, understand why Easter wanders about, and so capriciously, as it appears, through a space of five weeks; and, besides this, its fluctuation, affecting the whole period of Lent, causes much inconvenience in reference to many of the relations of every-day civil life. That the Church had the *power* to make the change is admitted by *Clavius* himself (Cap. I., § 3); and he also tells us that there were persons at that time who urged the change. But he defends the traditional usage, on the ground that it was the immemorial practice of the Church, sanctioned by Popes and Councils from the earliest times: which practice was founded on the desire of the Primitive Church to keep the Christian Pascha as nearly as possible at the same time as the Jewish Passover (which was its type), and, therefore, to observe the same conditions as regards the Vernal Equinox, and 14th day of the Moon. ([2])

Had Easter been always an immoveable Feast, like Christmas Day, many and bitter controversies would have been spared to the Church; diversities of practice in its observance would not have occurred; and much perplexity and trouble would have been avoided in devising means whereby the difficulties of a Luni-Solar Easter might be overcome. It is not likely that, in the present state of Western Christendom, all Churches will ever agree in making this important change. But, were it effected, the determination of the Festival of Easter would be most simple, as it would depend solely on the Sunday Letter of the year. It is needless to say that this change has been strongly advocated by all the best modern writers on the Calendar—Ideler, Delambre, De Morgan. John Bernouilli wrote a memoir on the subject to the Senate of Basle in 1724 (Op. Tom. iv., p. 494), in which he strongly recommends the same course. He points out that even if the Calendar gave (as it does *not*) the most exact determination of the Equinox and the Paschal Full Moon, yet that the universal Church could not keep Easter on the *same* day, by reason of the difference of time resulting from the difference of longitude (*vid. infra*).

Another advantage of fixing Easter would be that the Festival would be always kept nearly at the same time at which it first actually occurred, and so would be a real anniversary of the great event which it commemorates, whereas it now may, and often does, differ from it by five weeks.

However, such a change now is obviously impossible. The Roman Church would certainly not consent to it, and a schism in Christendom on a question so vitally affecting the practical affairs of civil life would be in the highest degree inconvenient. The present practice is regulated by Rules and Tables, which, however difficult and troublesome to ascertain and calculate at first, are now very easily applied. We have seen that some of the German Protestant States, which refused to adopt the Gregorian Correction and Rules, proposed to substitute an *astronomical* Easter for that of the *Calendar;* that is to say, to determine Easter by the true or mean Full Moon, instead of the moon of the 19 year Cycle. But this is obviously incompatible with the condition of always keeping Easter on the *same* Sunday all over the world; on the existing plan this is practically secured. Because, as the diurnal course of the sun is completed in 24 hours, every place on the circumference of the earth, where the sun rises and sets will, in the course of the same 24 hours (νυχθήμερον) have the sun successively on its meridian, and therefore some portion of the 24 hours assigned to Easter Day will be common to all. But, if the moment of conjunction determined by the true or mean moon be used, it must inevitably happen that Easter will sometimes be held on different Sundays, even in places very close to each other. For the time of New Moon will then be determined by the moment at which the centres of the earth, the sun, and the true (or mean) moon, are in one straight line. Suppose the *mean* moon to be used. Suppose also that the mean New Moon at London falls at 11 o'clock on Sunday night. In places two hours of longitude (30°) eastward of London, it is then 1 o'clock on Monday morning. Accordingly, Sunday is the first day of the moon at London, and Monday at those other places. The 14th day of the moon is, at London, on Saturday, and Easter Day will be the next day. At these and other places, the 14th moon is as Sunday, and Easter Day will be on the following Sunday, a week later than the former. Again, suppose the *true* new moon be taken. In the case just supposed, the same result will happen, of a difference of a week in the keeping of Easter. Indeed the Lunar Tables, whether for the mean or true motion of the moon, are sufficiently accurate to make it possible that an astronomical Easter determined by the mean or true Full Moon should be kept on one Sunday in St. Paul's Cathedral, and in Westminster Abbey on the following Sunday; for the difference of longitude between those two churches being about 7 seconds (St. Paul's the more eastward), Sunday morning begins at St. Paul's about seven seconds before it begins at the Abbey. Now, suppose Easter to be determined by the true Full Moon, and that on a *Saturday* evening at the Abbey the Paschal Full Moon happens at four seconds before midnight; then, at St. Paul's, it will happen three seconds after midnight, on *Sunday* morning. The result will be that at the Abbey this Sunday will be Easter Day; while, at St. Paul's, the Paschal Moon falling *on* this Sunday, the next Sunday will be Easter Day. Hence it is plain that *perfect astronomical* accuracy in the determination of Easter is *practically* impossible; and that if the moon be employed at all in its determination, it must be the moon of a *Cycle*, so arranged, however, as to keep as nearly as possible to the mean and true moons. And such is the principle on which the reformed Calendar of Gregory XIII. proceeds; though, from the state of astronomical knowledge at the time of that reformation, the greatest amount of possible accuracy was not attained.

It can only rarely happen that our Easter Day is a perfect anniversary of the event commemorated. The year of the Crucifixion, and, therefore, of the Resurrection is still an unsettled point. The ancient authorities generally assign it to A.D. 29, but assuming with Ussher that it happened A.D. 33, and that the Crucifixion (which took place on Friday) fell on April 3rd, it follows that the Resurrection occurred on April 5th. Now, this is only one out of the 35 days on which Easter may fall.

If, with Browne (*Ordo Sæclorum*, p. 55), the Crucifixion be assigned to A.D. 29, Friday, March 18th, it would follow that the Resurrection took place on March 20th; which date has been from very early times outside the Paschal limits, and therefore no exact *anniversary* of the Resurrection has been kept in the Church.

(1) Epiphamus tells us that some of the Quartodecimans of Cappadocia always kept their Easter on the 25th of March, whatever day of the week or Lunar month it might be. Bede also informs us that the Christians of Gaul, in the time of Pope Victor, did the same.—*Vide* Bingham, B. xx., c. v., § 2.

(2) Clavius insists on adhering to the Hebrew rule of celebrating Easter "propter sacramentum et recondita mysteria, quæ in ejusmodi celebratione Paschæ resurrectionis Dominicæ includuntur:" and yet

the Christian Church did not hesitate to deviate from the Jewish usage, when it laid it down as a fundamental rule, that Easter must be celebrated on a *Sunday*, and *never on the* 14*th day of the moon*, even though it might be a Sunday.

152. The adoption of the Gregorian change of style in England occasioned to many persons much uneasiness and distress, on account of the Gregorian Easter following so often on a day different from the Old Style, or Julian Easter. However, there were some years in which both Easters coincided, and so must have proved a source of much consolation to them. Thus, between 1752 and 1800 there were eighteen years in which such coinciding took place. I propose now to go a little more into detail respecting the relation between the New and Old Easter.

The condition that there shall be a coincidence between the Julian and Gregorian Easter is (of course) that the former shall precede the latter by exactly as many days as the number due to the solar equation, *plus* the ten omitted in 1582. Thus, for the interval from 1583 to 1699, as the number of omitted days amounted to ten, if the Old and New Easter differed by ten nominal days, then they occurred on the same actual day, *e.g.*, A.D. 1600 the Old Easter fell on March 23, Old Style; that is, on April 2, New Style. But the New Easter that year fell on April 2; that is, both Easters coincided. In the same way, if in any year of the century 1700–1799, the two Easters differed by eleven nominal days, they fell on the same actual day; because the difference of style that century amounted to $10 + 1 = 11$ days. Thus, in the year 1706 the Old Style Easter fell on March 24, *i.e.*, adding 11, on April 4, New Style. But the Gregorian Easter fell that year on April 4. Consequently both Easters coincided. Similarly, in this present century, the difference of style being twelve, if in any year the Old and New Easters be separated by twelve nominal days, they fall on the same actual day. For example, in 1876 the Old Easter falls on April 3, and the New on April 16; and, therefore, being twelve nominal days apart, they actually coincide. And so on with respect to other succeeding centuries.

We may now investigate the subject more generally.

Referring to Articles 137, 142, and using the notation there employed, we have the date of the Gregorian Easter, or

$$\pi = P + L - \lambda. \qquad (1)$$

Where P and λ are functions of the Gregorian Epact (ϵ), P being the date in March of the 15th day of the Paschal Moon, λ the Calendar Letter of the 15th day of the moon, and L is the number of the Sunday Letter of the given year in the scale

A	B	C	D	E	F	G.
1	2	3	4	5	6	7.

Similarly (Art. 142) we have for the Julian, or Old Style, Easter that year

$$\pi' = P' + L' - \lambda'. \qquad (2)$$

Where P' and λ' are similar functions of the Julian Epact (ϵ'), and L', the Julian Sunday Letter of the same year, referred to the same scale.

As π' is the date of the Old Easter in the *Old Style reckoning*, in order to get its date in the *New Style reckoning* we must add to π' the ten days omitted in 1582 and the Solar Equation for the centuries subsequent to 1600, so that, in *New Style reckoning*,

$$\pi' = P' + L' - \lambda' + 10 + \odot. \qquad (3)$$

Hence the number of *actual* days intervening between the New Style Easter and the Old Style Easter is given, generally, by subtracting (3) from (1), or *vice versâ*, which gives

$$\pi - \pi' = P - P' + (L - \lambda) - (L' - \lambda') - (10 + \odot) \qquad (4)$$
$$= P - P' + (L - L') - (\lambda - \lambda') - (10 + \odot). \qquad (5)$$

Ex.—Find $\pi - \pi'$ for 1876.

Here Golden Number is XV., therefore by *the Expanded Table of Epacts* (Art. 123) or by the formula (Art. 136),

$$\epsilon = 4.$$

Also $$L = 1.$$

So by Equations (1) and (2), Art. 142,

$$\epsilon' = 12$$
$$L' = 3.$$

Therefore, as ϵ and ϵ' are each less than 24, we get

$$P = 45 - 4 = 41$$
$$\lambda = \left(\frac{27 - 11}{7}\right)_r = 2;\ L - \lambda = 1 - 2 = -1 = 6.$$

Hence $\pi = 41 + 6 = 47$ of March = 16 April, New Easter.

Again,

$$P' = 45 - 12 = 33.$$
$$\lambda' = \left(\frac{27 - 12}{7}\right)_r = 1 \therefore L' - \lambda' = 3 - 1 = 2, \text{ and } 10 + \odot = 12 \text{ in } 1800;$$
$$\therefore \pi' = 33 + 2 + 12 = \text{March } 47 = \text{April } 16, \text{ Old Easter}.$$

Therefore, in 1876 the New and Old Easter coincide.

The above may be found at once by means of Delambre's Table, Art. 144, when we know the Epacts ϵ and ϵ', and Sunday Letters A and C.

Look at Column V. (Letter A), and under $\epsilon = 4$, we find Easter Day π, April 16. Again, in Column VII. (C.), and under $\epsilon' = 12$, we find Easter Day π', April 4. The same may, of course, be found by Clavius' Table, Art. 134.

N.B.—*The Gregorian and Julian Epacts may both be found by inspection of the Expanded Table* (Art. 123), when we know the Golden Number of any year. Thus, for the year 1876, look for the Centurial Number 18 (Index C.), and in the corresponding line of Epacts, under Golden Number XV. we find 4; look then in the line of Epacts whose Index is P, which, as we have already seen (Art. 127), contains the *perpetual* line of Julian Epacts, and we find 12.

It will be observed that for the *whole century* 1800-1899, the difference between the Gregorian and Julian Epacts is 8; that is to say,

$$\epsilon' - \epsilon = 8.$$

153. Reverting to Art. 152, we see that Equation (5) is useful for showing the different intervals by which the Gregorian and Julian Easters (the latter reduced as before explained to New Style reckoning) may be separated from each other for one or more *centuries*, without reference to particular years. In the first place, $P - P' = \epsilon' - \epsilon$, because P and P' denote the 15th of the respective Paschal Moons, and ϵ and ϵ' denote the 14th days of these moons, and the Epacts are written in retrograde order, the larger P having the smaller ϵ and *v. v.* Again, as the Julian Cycle of Epacts is invariable, and any Gregorian Cycle of Epacts continues unchanged for at least a century, $\epsilon' - \epsilon$ is constant as long as ϵ does not change. The change is always in the direction of a *diminution* of the Gregorian Epact. Now, this difference can be found at once by inspection of the Expanded Table of Epacts. For when we know the *century*, we find on the same horizontal line with it the corresponding Cycle of Epacts; and Index P gives the invariable line of Julian Epacts; and comparing these two lines, we see that each Julian Epact differs from the corresponding Gregorian by a constant number; or, when the century is given, we can (without the Expanded Table) calculate by Equation (2) (Art. 136) the Gregorian Epact for any year of it, and then, by the known law of formation, calculate all the rest for that Cycle.

Thus, for example, we see by the Expanded Table of Epacts that the Julian Cycle of Epacts (Index (P)) *exceeds* the Gregorian line of Epacts for century 16 (Index D) by 7 units, and exceeds the Gregorian line for centuries 17 and 18 by 8 units, and so on. The same result, of course, follows from the formulæ

$$\epsilon' = \left(\frac{11\ N - 3}{30}\right)_r,$$

$$\epsilon = \left(\frac{11\ N - 10}{30}\right)_r - (\sigma - 16) + \left(\frac{\sigma - 16}{4}\right)_w + \left(\frac{\sigma - 15 - a}{3}\right)_w.$$

154. To find the intervals between Old and New Style Easter by means of Equation (4) of Art. 152, we must first consider the value of $P - P'$ in terms of ϵ and ϵ'. There are three cases to be considered:—

I. ϵ and ϵ' both > 23, or both < 24,

then
$$P - P' = \epsilon' - \epsilon.$$

II. $\epsilon < 24,\ \epsilon' > 23,$
$$P - P' = (\epsilon' - \epsilon) - 30.$$

III. $\epsilon > 23,\ \epsilon' < 24,$
$$P - P' = 30 + (\epsilon' - \epsilon).$$

It must also be borne in mind that $L - \lambda$ and $L' - \lambda'$ may each separately have any value from 0 to 6; but that neither can be negative. A good test of the accuracy of the result is to observe if $\pi - \pi'$ is, as it should be, a multiple of 7.

Ex. Find what was the difference between the two Easters in A.D. 1600.

Here we find
$$\epsilon = 15$$
$$\epsilon' = 22 \therefore P - P' = 7$$

$$\left.\begin{array}{l} \left.\begin{array}{l} L = 1 \\ \lambda = 5 \end{array}\right\} L - \lambda = -4 = 3 \\ \left.\begin{array}{l} L' = 5 \\ \lambda' = 5 \end{array}\right\} L' - \lambda' = 0 \end{array}\right\} \odot = 0.$$

Hence by (4)
$$\pi - \pi' = 7 + 3 - 0 - 10 = 0,$$
or the two Easters coincide.

Ex. 2. Find the difference of the two Easters A.D. 1602.

Here
$$\left.\begin{array}{l} \epsilon = 7 \\ \epsilon' = 14 \end{array}\right\} \begin{array}{l} L = 5\ \ \lambda = 5 \\ L' = 3\ \ \lambda' = 6 \end{array} \left\{\begin{array}{l} L - \lambda = 0, \\ L' - \lambda' = 4, \end{array}\right.$$

so $P - P' = 7$, and Equation (4) of Art. 152 becomes

$$\pi - \pi' = 7 + 0 - 4 - 10 = -7,$$

showing that New Style Easter happens a week before Old Style Easter, expressed in New Style reckoning.

Ex. 3. Find the difference in 1625.

Here $\left.\begin{matrix}\epsilon = 21\\ \epsilon' = 28\end{matrix}\right\}$, so $P - P' = \epsilon' - \epsilon - 30 = -23$ (Case II. of this Art.) $\left.\begin{matrix}L = 5 & \lambda = 6\\ L' = 2 & \lambda' = 1\end{matrix}\right\}$

$\therefore \left.\begin{matrix}L - \lambda = 6\\ L' - \lambda' = 1\end{matrix}\right\}$,

So $$\pi - \pi' = -23 + 6 - 1 - 10 = -28,$$

showing that New Easter falls four weeks before the Old.

Ex. 4. Find the difference in 1603.

$$\left.\begin{matrix}\epsilon = 18\\ \epsilon' = 25\end{matrix}\right\} \left.\begin{matrix}L = 5 & \lambda = 2\\ L' = 2 & \lambda' = 4\end{matrix}\right\} \left.\begin{matrix}L - \lambda = 3\\ L' - \lambda' = 5\end{matrix}\right\}$$

$$P - P' = +7 - 30 \text{ (Case II.)} = -23,$$

and $$\pi - \pi' = -23 + 3 - 5 - 10 = -35,$$

showing that New Easter falls five weeks before the Old.

In the above Examples we have instances of the *coincidence* of the two Easters and of the respective intervals of *one*, *four*, and *five* weeks between them.

There are examples of each of these four cases also in the interval between 1583 and 1599; *e.g.*, 1583 (0); 1586 (7); 1587 (28); 1595 (35).

In the interval between 1583 and 1699, both inclusive, there were

53	cases of coincidence,		
39	„	*one* week's interval,	
7	„	*four* weeks'	„
18	„	*five* „	„

For those who adopt the Gregorian reckoning, no true Easter can fall after April 25, New Style. Consequently, during the interval just mentioned, when the difference of style was 10 days, every Old Easter that fell after April 15, Old Style, was not a true Easter.

155. The special advantage of Equation (5), Art. 152, is that it enables us, knowing only the *century*, to find all the possible differences of the two Easters without going through the different years of the century. We may take $P - P' = \epsilon' - \epsilon$, provided we notice that, on comparing any two lines of Epacts in the Expanded Table, if ϵ' and ϵ be any corresponding pair (*i. e.*, in the same vertical column), $\epsilon' - \epsilon$ has two different values —one positive, the other negative—such that positive value = 30 + negative value. This being presumed, we observe that $P - P'$ and $\odot + 10$ are common to the whole century, for $\epsilon' - \epsilon$ remains constant during the century. Also $L - L' = \left(\frac{10 + \odot}{7}\right)_r$

(Art. 94) also depends only on the century. Again, $\lambda - \lambda' = \left(\frac{\epsilon' - \epsilon}{7}\right)_r$ (where both values of $\epsilon' - \epsilon$ are to be considered, and also the result of adding or subtracting a multiple of 7), which again only depends on σ, the centurial year.

Since $\pi - \pi'$ must be a multiple of 7, such values of $L - L' - (\lambda - \lambda')$ must be excluded as will fail to ensure this result. Further, $L - L' - (\lambda - \lambda')$ cannot exceed 6, for it equals $L - \lambda - (L' - \lambda')$ (Art. 154). All this is illustrated by an Example.

Ex. 1. Find the values of $\pi - \pi'$ for the interval 1583–1699.

$$P - P' = \epsilon' - \epsilon = 7, \text{ or } -23 \text{ (Art. 154)}; \ \odot = 0$$

$$L - L' = \left(\frac{10 + \odot}{7}\right)_r \text{ (Art. 94} = \left(\frac{10}{7}\right)_r = 3 \text{ or } -4$$

$$\lambda - \lambda' = \left(\frac{7}{7}\right)_r = 0, \text{ or } \left(-\frac{23}{7}\right)_r = \left(\frac{28-23}{7}\right)_r = 5 \text{ or } -2.$$

Substituting these values in (5), we get for

I. $\epsilon' - \epsilon = 7$

$$(a).\ \pi - \pi' = 7 + 3 - 0 - 10 = 0$$
$$(b).\ \pi - \pi' = 7 - 4 - 0 - 10 = -7.$$

II. $\epsilon' - \epsilon = -23$

$$(c).\ \pi - \pi' = -23 + 3 + 2 - 10 = -28$$
$$(d).\ \pi - \pi' = -23 + 3 - 5 - 10 = -35.$$

These are the only admissible combinations of $L - L'$ and $\lambda - \lambda'$.

Hence we see that there can be but *four* values of $\pi - \pi'$ for the interval just indicated, viz., 0, – 7, – 28, – 35.

156. The results arrived at in the last Article may be put in this way:—

We have
$$\pi - \pi' = \left.\begin{matrix}7\\-23\end{matrix}\right\} + L - \lambda - (L' - \lambda') - 10$$
$$= \left.\begin{matrix}7\\-23\end{matrix}\right\} + L - L' - (\lambda - \lambda') - 10.$$

I. If $\epsilon' - \epsilon = 7$ we have two cases—

(*a*). If the Old Easter Day be nearer to its 15th moon than the New Easter Day is to its 15th moon; in other words, if $L' - \lambda'$ is less than $L - \lambda$, then π will be more than 7 days before π' (*the Old Easter being reckoned in Old Style*): and when the 10 days are added, both must coincide.

(*b*). But if $L' - \lambda' > L - \lambda$, then the two Easters (the New Easter reckoned in New Style, the Old Easter in Old Style) will be less than 7 days asunder, and when the 10 days are added to π', it will be 7 days in advance of π: or $\pi - \pi' = -7$.

II. If $\epsilon' - \epsilon = -23$, or the Old Paschal Full Moon be 23 days later than the New, then:—

(*c*). If $L - \lambda$ be $> L' - \lambda'$, the two Easters will be *less* than 23 days apart, and therefore when the 10 is added, we must have

$$\pi' - \pi = 28.$$

(*d*). If $L - \lambda < L' - \lambda'$, the two Easters are *more* than 23 days apart, and so when 10 days are added to π', we have

$$\pi' - \pi = 35.$$

157. Let us now inquire what are the differences between the two Easters for the century 1700–1799.

Here
$$\epsilon' - \epsilon = 8, \text{ or } -22.$$

$$L - L' = \left(\frac{10 + \odot}{7}\right)_r = 4, -3$$

$$\lambda - \lambda' = \left(\frac{8}{7}\right)_r = 1 \text{ or } -6$$

or
$$= \left(-\frac{22}{7}\right)_r = \left(\frac{6}{7}\right)_r = -1 \text{ or } 6.$$

Now, the condition $L - L' - \overline{\lambda - \lambda'}$ not > 6 or < -6 gives as possible values of this quantity $3, 5, -4, -2$.

Hence,

I. $\epsilon' - \epsilon = 8$,

$$\pi - \pi' = 8 - 11 + L - L' - \overline{\lambda - \lambda'}$$

$$= -3 \begin{matrix} +3 \\ +5 \\ -4 \\ -2 \end{matrix} = 0 \text{ or } -7, \text{ since } \pi - \pi' = \text{mult. } 7.$$

II. $\epsilon' - \epsilon = -22$,

$$\therefore \pi - \pi' = -33 \begin{matrix} +3 \\ +5 \\ -4 \\ -2 \end{matrix} = -28, \text{ or } -35.$$

Hence in this century, as in the preceding, the Easters may coincide, or be separated by one, four, or five weeks, the Old Easter (reckoned New Style) falling *after* the New.

In this century, according to the Gregorians, no Old Easter falling after April 14 (Old Style) was a true Easter.

158. Let us see what are the differences between the Easters for the century 1800 to 1899.

$$\epsilon - \epsilon = 8 \text{ or } -22 \quad 10 + \odot = 12,$$

$$L - L' = \left(\frac{12}{7}\right)_r = 5 \text{ or } -2,$$

$$\lambda - \lambda' = \left(\frac{8}{7}\right)_r = 1 \text{ or } -6, \text{ or } = \frac{-22}{7} = 6 \text{ or } -1,$$

so

$$L - L' - \overline{\lambda - \lambda'} = 6, 4, 3, -1.$$

Hence

I. $\epsilon' - \epsilon = 8$,

$$\pi - \pi' = 8 - 12 + L - L' - \overline{\lambda - \lambda'}$$

$$= -4 \begin{matrix} +6 \\ +4 \\ -3 \\ -1 \end{matrix} = 0 \text{ or } -7$$

II. $\epsilon' - \epsilon = -22$,

$$\pi - \pi' = -34 \begin{matrix} +6 \\ +4 \\ -3 \\ -1 \end{matrix} = -28 \text{ or } -35.$$

The same will be found true for 1900–1999, and for 2000–2099. In 2092, 28 occurs for the *last time*. In other words, the last occasion on which the Old and New Easter will differ by four weeks will be A.D. 2092.

In the present century, since $\odot + 10 = 12$, any Old Style Easter falling after April 13 is not legitimate. The number of coincidences this century will be 34. There was one in 1876; the next will be in 1879.

159. Find the differences of the Easters for the century 2100–2199.

Here $\epsilon' - \epsilon = 9$, and -21: $10 + \odot = 14$, $L - L' = \left(\frac{14}{7}\right)_r = 0$

$$\lambda - \lambda' = \left(\frac{9}{7}\right)_r = 2 \text{ or } -5, \text{ or } = \left(\frac{-21}{7}\right)_r = 0 \therefore L - L' - \overline{\lambda - \lambda'} = 0, 5, -2.$$

Hence,

I.

$$\epsilon' - \epsilon = 9, \therefore \pi - \pi' = -5 \begin{matrix} +0 \\ -2 \\ +5 \end{matrix} = 0 \text{ or } -7.$$

II.

$$\epsilon' - \epsilon = -21, \therefore \pi - \pi' = -35 \begin{matrix} +0 \\ -2 \\ +5 \end{matrix} = -35.$$

Thus the only values of $\pi - \pi'$ in this century are 0, -7, -35; and this will continue down to 2436, inclusive. To find value of $\pi - \pi'$ in 2437, we revert to method of Art. 154, and find

$$\epsilon = 23 \quad L - \lambda = 0$$
$$\epsilon' = 3 \quad L' - \lambda' = 6$$

$\therefore \pi = 22 + 0 = 22$

$\pi' = 42 + 6 = 48$ (Old Style), *i.e.*, adding $10 + \quad = 16$, $\pi' = 64$ (New Style).

Hence $$\pi - \pi' = -42.$$

160. The series 0, 7, 35, 42 will continue up to 2698, when 0 *occurs for the last time.* In other words, *the New and Old Easters will coincide for the last time* in 2698.

That year both will fall on April 24, New Style.

It is easy to show from the General Equation (4) (Art. 152) that century 26 is the last in which $\pi - \pi'$ can $= 0$.

In order that the difference between the two Easters (both reckoned in New Style) may vanish, it is obviously necessary and sufficient that the Old Easter $(\pi)'$, reckoned in Old Style, should fall *before* the New Easter (π); so that when the omitted days $(10 + \quad)$ are added to π' (to reduce it to New Style), the difference $\pi - \pi'$ may become 0.

Taking Equation (4) of Art. 152, we require

$$\epsilon - \epsilon' - (10 + \odot) + \rho = 0, \text{ where } \rho = L - \lambda - (L' - \lambda').$$

That this may be possible, we must obviously take the *positive* values of $\epsilon' - \epsilon$ and ρ for ρ is never greater than 6.

Let us see how the terms $\epsilon' - \epsilon - (10 + \odot)$ vary in successive centuries.

From 1700–1799,	$\epsilon' - \epsilon - (10 + \odot)$		$= -3.$
„	1800–2199,	„	„	$= -4.$
„	2200–2399,	„	„	$= -5.$
„	2400–2699,	„	„	$= -6.$

After this the quantity in question becomes less than -6, and as ρ cannot be > 6, we see that the last year in which $\pi - \pi'$ can vanish is the year 2699.

Strictly speaking, 2698 is the last year in which $\pi - \pi' = 0$, for in 2699, ρ is not equal to 6.

161. The Prayer Book definition of Easter Day.

Before the Revision of 1662, no definition of Easter Day was given in the English Prayer Book. The definition then added was probably drawn up by Bishop Cosin, one of the Committee of Revision, as it agrees very closely, though not exactly, with the definition given by him in his "Private Devotions," published several years before (1627). The following are the words of the Prayer Book of 1662:—"EASTER DAY is always the first Sunday after the first Full Moon which happens next after the one and twentieth day of March. And if the Full Moon happens upon a Sunday, *Easter Day* is the Sunday after." This definition is obviously erroneous, because according to it Easter Day could never fall before March 23; whereas, according to the perpetual usage of the Church it might fall on March 22. And the error is the more remarkable if, as I have just suggested, Bishop Cosin was the author, inasmuch as in his "Private Devotions" he does not fall into it. In fact, the definition given in the sixth edition of that work, published before the Revision of 1662, agrees exactly with the definition in our present Prayer Book, which dates from the year 1752. The Church of England, in common with several of the Continental Protestant Churches, refused for a considerable time to adopt the Gregorian reformation of the Calendar, which by the Pope's Bull became the law of the Church of Rome in 1582. At length, the inconvenience of adhering to the old Church Calendar became so great that an Act of Parliament (24 Geo. II., c. 23) was passed in the year 1751 for regulating the commencement of the year, and for correcting the Calendar then in use. By this statute the definition of Easter Day, as given in the Prayer Book of 1662, was amended; the old method of computing the Easter Full Moons was abolished; as also the "Table to find Easter for ever," founded thereon. New Tables and Rules, specially prepared under the direction of the then Astronomer Royal, Dr. Bradley, were annexed to the statute, "for the fixing the true time of the celebration of the Feast of Easter, and the finding the times of the Full Moons on which the same depended, so as the same should agree as nearly as might be with the Decree of the General Council of Nice, and also with the practice of foreign countries." Thus the Gregorian reformation of the Calendar was at last adopted in England; and Tables were constructed and inserted in the Prayer Book, for the determination of Easter Day in future; the chief difference between them and the corresponding Gregorian Tables consisting in this, that in the English Tables the old Golden Numbers were retained for denoting the Ecclesiastical New Moons, while in the Gregorian Tables a different set of numbers, connected by a certain relation with the Golden Numbers, called Epacts, were employed for that purpose.

162. The definition of Easter in the Statute of Geo. II. and our present Prayer Books is as follows:—

"EASTER DAY is always the first Sunday after the Full Moon which happens upon, or next after, the twenty-first day of March; and if the Full Moon happens upon a Sunday, *Easter Day* is the Sunday after."

The only real objection to this definition—and it is a slight one—is this, that the second clause of it is superfluous, since by the first clause Easter Day must be the Sunday *after* the Full Moon. But the late Professor De Morgan has found fault with it on a different ground, in a valuable paper *On the Ecclesiastical Calendar*, contributed by him to the "Companion to the British Almanac for 1845," and in several subsequent publications of his.

163. I shall state his objection in his own words (Compan. to Almanac, p. 33):—

"1. The law which regulates Easter in Great Britain declares that whenever the Full Moon on or next after March 21 falls on a Sunday, that Sunday is not Easter Sunday, but the next; it also prescribes Rules for determining Easter.

"2. In defiance of the precept, though in accordance with the Rules, the Easter Sunday of 1845 was on the very day of the Full Moon next following March 21.

"3. One part of the reason of this [discrepancy] is that the British Legislature misunderstood the definition of Easter used in the [Gregorian] Rules which they adopted, thinking that it depended upon the *Full Moon*, whereas it depends upon the *fourteenth day* of the moon, the day of the New Moon being counted as the *first*. Now, Full Moon never happens before the *fifteenth* day of this reckoning.

"4. The other part of the reason of this discrepancy is that the Legislature supposed the moon of the Calendar to be the same as the moon of the heavens, which neither is nor was intended to be the case: the moon of the Calendar being made to vary from the moon of the heavens not only for convenience of calculation, but also to prevent Easter Day from falling on the day of the Jewish Passover.

"5. These two errors very often compensate one another; for though the fourteenth day is very often a day behind the Calendar Full Moon, yet the Calendar Moon is also very often a day before the real moon, so that the fourteenth day of Calendar Moon is frequently the day of the real Full Moon. But they [the errors] do not always do so; and it should never be matter of surprise, if Easter fell on the Sunday of the Full Moon, whether real or Calendar.

"6. It is not correct to say that Easter was made to fall wrongly in 1845: it fell where the legislators, who correctly copied the Rule of the Roman Church, intended it should fall, though they did not correctly give [in the definition] the explanation of the Rule they intended to use."

Now, in reply to all this, it is sufficient to say that there is no reason whatever to believe that the expression "Full Moon" in the definition of Easter was intended to denote the *real* Moon of the heavens, or the astronomical Mean Moon. Such a confusion would betray a total ignorance of the fundamental principles on which the ancient Church Calendar was constructed, and which were retained and perpetually insisted on by Clavius in his great work on the Gregorian Calendar—such an ignorance as we can hardly conceive in the case of Bishop Cosin, one of the most learned ritualists of the age, and with whose definition of Easter Day the definition in the Act of Parliament agrees *verbatim*. Nor can it be supposed that the framers of the Tables and Rules contained in the Act, who evidently studied most carefully the Gregorian Tables, neglected to make themselves acquainted with the true definition of Easter itself, the central point of the whole subject, and one which Clavius took such pains to explain in the passages of his work quoted by De Morgan (pp. 17, 18), and in several other places of the same volume. So accurate and learned an astronomer as Dr. Bradley was not likely to have fallen into a confusion which, according to De Morgan himself, a little attention to Clavius' words might have avoided. Moreover, the Breviary and Missal of the Church of Rome contained a carefully and clearly drawn up chapter on the very subject of the reformed Calendar, and specially of the Festival of Easter. This source of information was easily accessible, and would naturally have been consulted by Dr. Bradley and his fellow labourers. These books give the definition of Easter in a form which admits of no misconception:—"Ex decreto sacri Concilii Niceni Pascha, ex quo reliqua Festa mobilia pendent, celebrari debet die Dominico qui proximè succedit XIV. Lunæ primi mensis; is vero apud Hebræos vocatur primus mensis, cujus XIV. Luna vel cadit in diem Verni Æquinoctii, quod die 21 mensis Martii contingit, vel propius ipsum sequitur." And, lastly, there seems to be very clear evidence on the face of the Act itself, that no such misconception as De Morgan supposes existed in the minds of the framers of the Act respecting the definition of Easter. On the contrary, they very pointedly distinguish between the *real* and *Ecclesiastical* (or Calendar) Full Moon. In the explanation of the "Table to find Easter from the year 1900 to 2199, inclusive," we read as follows:—"The Golden Numbers in the foregoing Calendar will point out the days of the Paschal Full Moons till the year of our Lord 1900; at which time, in order that the *Ecclesiastical Full Moons* may fall nearly on the same days with the *real Full Moons*, the Golden Numbers must be removed to different days of the Calendar, &c." De Morgan himself could hardly distinguish more expressly between the *real* and *Ecclesiastical* Full Moon.

Now, what was intended by the expression "*Ecclesiastical* Full Moon" in the above explanation? Beyond all doubt, the fourteenth day of the Calendar or Ecclesiastical Moon, reckoned from the day of the New Moon inclusive, these New Moons themselves being denoted by the Golden Numbers. This was the well-known and received usage of Church writers when speaking of Easter. The following are the words of one of the highest modern authorities on the subject (Ideler, *Handbuch der Chronologie*, ii., 198). After explaining the Calendar New Moons and the Golden Numbers, he adds, "From the New Moons we must now further derive the *Full Moons*. Everywhere in the treatises on the Festival of Easter in the works of the Church writers we find the expression τεσσαρεσκαιδεκάτη (ἡμέρα τῆς σελήνης), or *Luna decima quarta*, used to denote the day of the Full Moon." And he accounts for it thus:—"The real Full Moon happens on the average about fifteen days after conjunction; but the ancient Greeks, who made use of the actual Lunar Months, reckoned the age of the moon not as we do, from its conjunction with the sun, but from the time of its thin crescent becoming visible in the evening sky, from which time they began their Lunar month. Now, since from this first phase to the Full Moon, thirteen days usually elapsed, the first calculators of Easter, in order to arrive from the new to the full light of the moon, counted thirteen, or, including the day of the New Moon (Νουμηνία), fourteen days forwards." When the Greeks subsequently made use of the moon of the 19-year cycle, instead of the actual moon, to determine Easter Day, they retained the old usage of the fourteenth day to denote the cyclic Full Moon, as it had originally denoted the real Full Moon (*ib.* i., 262). Hence it was that, in the definition of Easter in Bishop Cosin's "Devotions," and in the Prayer Book of 1662, the phrase "Full Moon" was used, not to denote the real Full Moon, but the Ecclesiastical Full Moon—in other words, the fourteenth day of the Calendar, or Cyclic Moon. A striking and instructive instance of this usage occurs in the remarkably able paper by Lord Macclesfield, published in the 46th Vol. of the Philos. Transact., pp. 417, *sq.*, the year (1750) before the Act of Parliament in question was passed. In this paper the writer strongly advocates the adoption of the Gregorian reformation in England. Comparing the old and new Calendars, he says that the Church of England and the Church of Rome still agree in this, that both of them mark (the former by the Golden Numbers, the latter by the Epacts corresponding to them) the days on which their *Ecclesiastical* New Moons are supposed to happen; and that in both alike the *fourteenth day* of the moon, inclusive, or that *Full Moon* which falls upon or next after the 21st of March, is the *Paschal limit or* (*Paschal*) *Full Moon;* and the Sunday next following that Full Moon is by both celebrated as Easter Day. It can hardly be doubted that Dr. Bradley was acquainted with this paper; and, if so, he could hardly have fallen into the confusion with which De Morgan charges him, a confusion so carefully guarded against by Lord Macclesfield.

There seems, then, to be no valid reason for, but many against, supposing, with De Morgan, that the use of the expression "Full Moon" in the present Prayer Book definition of Easter Day was a mistake arising from a misapprehension on the part of the framers of the statute of Geo. II., respecting the way in which the moon was employed to determine that Festival. They corrected the definition of Easter as they found it in the Prayer Book of 1662, so far as it really needed correction. They left the expression "Full Moon" unchanged, because it was, in the immemorial usage of the Church, the recognised mode of denoting the fourteenth day of the *Calendar Moon*. They have guarded against any misapprehension of their meaning by expressly distinguishing between the *Ecclesiastical* Full Moons and the *real* Full Moons. The Tables which they give for finding Easter Day—both the temporary and the general Tables—all are strictly in accordance with the above sense of the Paschal Full Moon, as denoting the fourteenth day reckoned from the Paschal New Moon inclusive; while the said Tables are inconsistent with the definition, if "Full Moon" be interpreted as De Morgan does. He persuaded himself that he detected a paradox, which seems to have no existence but in his own ingenious mind.

On the other hand, it is equally certain that the definition has been misunderstood in the way contended for by De Morgan. In the year 1818 the real Full Moon happened on Sunday, March 22, and that day was kept as Easter Day. Much discussion arose on the subject, and many protests were made against what was thought to be a direct contradiction of the second clause of the Prayer Book definition, viz., "if the Full Moon happens upon a Sunday, Easter Day is the Sunday after." It was, of course, supposed that the "Full Moon" of the definition denoted the real Full Moon. No explanation of the seeming contradiction was then given. But the difficulty at once disappears when it is remembered that the "Full Moon" is *not* the actual Full Moon, but the fourteenth day of the Calendar moon, which that year fell on March 21, Saturday; so that Easter Day was rightly kept on the next day, the 22nd. Similarly, in the year 1845 the real Full Moon fell on March 23, and yet the Tables directed that Easter should be kept on that day; and rightly, because the fourteenth day of the Calendar moon fell on Saturday, the 22nd, whereas had "Full Moon" meant the actual moon, Easter should, in accordance with the second clause of the definition, have been kept on the 30th. It was on this occasion that De Morgan wrote the paper already referred to, in which he charged the authors of the definition with the confusion which really belonged to those who did not understand the technical language of the Church Calendar.

However, as the definition was then misunderstood, and may be so again, it seems desirable that an explanatory note should be appended to it, to the following effect:—

NOTE.—*That the moon referred to in this Rule is not the actual moon of the heavens, but the moon of the Ecclesiastical Calendar, which is to be taken as full on its fourteenth day, the day of the Ecclesiastical New Moon being counted as the first day of the moon.*

164. It may be further remarked here, that this difference between the Astronomical and Calendar moon excited much attention in Germany in the year 1724. The Protestant states in Germany refused to adopt the Gregorian Calendar, partly because it emanated from Rome, and they objected to the *mandamus* of the Papal Bull; and partly because the new Calendar did not agree accurately with the actual phenomena.

Hence, in Germany, the Old Julian Calendar continued in use along with the Gregorian, and a diversity of usage between the two religious parties necessarily arose in reference to the moveable Festivals. The great inconveniences which arose induced the Evangelical states, in the year 1699, to adopt a third Calendar, different from both the others. They so far agreed with the Gregorian as to adopt the correction in regard to the Equinox, dropping eleven days in the year 1700, and passing at once from February 18th to March 1st. But in the determination of *Easter* and the other moveable Festivals they substituted astronomical calculations for the Calendar Cycle. And as their common Easter Meridian, they fixed on that of Uraneburg, the place where Tycho Brahe had his Observatory. For some time this astronomical calculation of Easter agreed with the date given by the Gregorian Calendar. But at last, in 1724, a difference arose, the Festival falling on 16th of April according to the latter, while the former assigned the 9th as the day. Very serious disturbances ensued between the Protestants and the Roman Catholics. It was upon this occasion that Bernouilli wrote the memoir before alluded to (Art. 150). In 1744 a similar diversity occurred, and the Diet had to interfere to settle the dispute. At last, in 1788, when the difference again took place, the Calendar of 1699 was given up by the *Corpus Evangelicorum*, and it was decided that Easter should be henceforth observed according to the Rules of the Gregorian Calendar.

165. *The Thirty-five Ecclesiastical Calendars.*

Easter Day may fall (as we have seen) on 35 different days, viz., from March 22 to April 25, both inclusive. All the other moveable Festivals depending on it will of course have the same range of incidence. If, then, there were constructed for *each* of those 35 years a Calendar, or Almanac, exhibiting the moveable Festivals and other Holidays which occur throughout the course of the Church year, we should have a complete set of Ecclesiastical Calendars, representing all the possible varieties that can occur in the Church year. And for any given year, whether in Old Style or New, we should, by means of a suitable Index, be able to find the Church Calendar belonging to that year. This work has been done by several writers on the Ecclesiastical Calendar. The 35 Calendars are found very completely and elaborately set forth in Gavantus' well-known work (Thesaurus Sacr. Rit., edit. Merati, Ven., 1792), also in the earlier editions of "L'Art de Verifier les Dates." Recently these 35 Calendars have been published in a more accessible and otherwise more convenient form—in France, by Francœur, and in England by Professor de Morgan. They are both, the latter especially, very useful manuals. I shall content myself with a short general description of these Almanacs, with special reference to De Morgan's work.

Number 1 of the 35 Almanacs exhibits the days on which all the moveable and other Festivals fall in any year in which Easter Day occurs on March 22. Almanac 2 does the same for any year in which Easter Day falls on the next earliest day, March 23. Similarly, Almanac 3 relates to every year in which Easter Day falls on March 24. And so on to Almanac 35, which is to be used whenever Easter Day falls on April 25 (the latest day). The *Sundays* in any particular year depend on the Dominical Letter; and *Easter Sunday* requires further for its determination the Golden Number, or the Epact, of the year. Now, we know that in the Julian unreformed Calendar, the G. N. of a year in which Easter falls on March 22 must be XVI. (Art. 62), and in the Gregorian Calendar the Epact of such a year must be 23 (Art. 134). Hence, in any year *Old Style* that has for G. N. XVI. and Dom. Lett. D, or in any year New Style that has for its Epact 23 and Dom. Lett. D, Easter Day will fall on March 22, and Almanac 1 will be the Almanac for that year. Again, in any year Old Style that has G. N. XVI. or V., with Dom. Lett. E (Art. 62), or any year New Style that has Epact 23 or 22 (Art. 134), with Dom. Lett. E, Easter Day will fall on March 23; and, accordingly, Almanac 2 will be the Almanac for that year. Similarly, in any year Old Style which has for its Golden Number either XVI., V. or XIII., and Dominical Letter G, or in any year New Style which has for Epact 23, 22, 21, or 20, and Dom. Lett. G, Easter Day will fall on March 25, and Almanac 4 will be its Almanac; and so on to the end of the 35 Almanacs. Neither Francœur nor De Morgan makes use of the Golden Numbers; but instead of them they employ the corresponding *Dionysian* Epacts (Art. 113), which are related, as we have seen, to the Golden Numbers in the following manner:—

Golden Number.	I.,	II.,	III.,	IV.,	V.,	VI.,	VII.,	VIII.,	IX.,	X.,	XI.,	XII.,	XIII.,	XIV.,	XV.,	XVI.,	XVII.,	XVIII.,	XIX.
Dionys. Epact	0 / 30	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18.

Thus, for example, in Almanac 5, the New Style Epacts are 23, 22, 21, 20, 19, with Dom. Lett. A; while the Old Style Epacts are 15, 14, 12, 11, which, according to the series just given, correspond respectively to the G. Nos. XVI., V., XIII., II.

In making use of these Almanacs for *Leap* years, the following Rule must be observed: "When a year is Bissextile, in place of the months of January and February of the Almanac corresponding to that year, we must take these two months from the next following Almanac, adding a 29th day to February." The reason is that a Bissextile year has two Dom. Letters, in the *inverse* order; of which the *second* is to be used from 1st March to the end of the year, while the *first* serves for January and February. And as the 35 Almanacs proceed in the direct order of the Calendar Letters from D (March 22) onwards, it follows that the first of the two Dom. Letters of any Bissextile year is the same as the Dom. Letter of the next Almanac: *e.g.*, the Dom. Letters of A.D. 232 are A, G. The Almanac (as we shall presently show) corresponding to that year is 4; the Letter of which is G, because Easter Day in that Almanac falls on March 25, whose Calendar Letter is G. The Letter of Almanac 5 is A, which is the same as the first of the two Bissextile Letters in question.

166. As a specimen of those 35 Almanacs, I give below the first four months of Almanacs 1 and 9.

In Almanac 1, the heading "Old Style $\binom{\text{XVI}}{15}$ D 1" is the *Index* mark of the Table, and expresses that in the unreformed Calendar Easter Day fell on the 22nd of March, for any year A.D. prior to 1583, whenever the Golden Number XVI., or the Dionysian Epact 15, concurred with the Sunday Letter D; and that the corresponding Almanac is Almanac 1. Similarly, the heading "New Style (23) D 1" is the *Index* mark for all years subsequent to 1582, denoting that, in the Gregorian Calendar, Easter Day falls on the 22nd of March, on all years whose *Gregorian* Epact is 23 and Sunday Letter D. As we have before seen, these are the *only* combinations in the Old and New Styles, respectively, that will give Easter Day on the 22nd of March.

In Almanac 9, which is constructed for Easter Day falling on March 30th (8 days later than March 22nd), the Index mark "Old Style $\begin{pmatrix}\text{XIII.,} & \text{II.,} & \text{X.,} & \text{XVIII.}\\ 12, & 11, & 9, & 7\end{pmatrix}$ E 9" denotes that in the unreformed Julian Calendar Easter Day fell on March 30th in all years when any of the four Golden Nos. XIII., II., X., XVIII., or the corresponding Dionysian Epacts 12, 11, 9, 7, concurred with Sunday Letter E. And the Index mark "New Style (21, 20, 19, 18, 17, 16, 15,) E 9" similarly denotes that in the Gregorian Calendar Easter Day falls on March 30th whenever any one of the seven *Gregorian* Epacts 21, 20, 19, 18, 17, 16, 15, concurs with the Sunday Letter E. The "Index" referred to will be given and explained presently.

ALMANAC 1.

Old Style $\binom{\text{XVI}}{15}$ D 1. New Style (23) D 1.

Days.	January.	Days.	February.	Days.	March.	Days.	April.
1	Circumcision.	1	S. *Quinquag.*	1	S. *Lent* 4.	1	
2		2	Purification.	2		2	
3		3		3		3	
4	S. *Christm.* 2.	4	Ash Wedn.	4		4	
5		5		5		5	S. *Easter* 2.
6	Epiphany.	6		6		6	
7		7		7		7	
8		8	S. *Lent* 1.	8	S. *Lent* 5.	8	
9		9		9		9	
10		10		10		10	
11	S. *Epiph.* 1.	11		11		11	
12		12		12		12	S. *Easter* 3.
13		13		13		13	
14		14		14		14	
15		15	S. *Lent* 2.	15	S. *Lent* 6, *Palm.*	15	
16		16		16		16	
17		17		17		17	
18	S. *Septuag.*	18		18		18	
19		19		19		19	S. *Easter* 4.
20		20		20		20	
21		21		21		21	
22		22	S. *Lent* 3.	22	S. *Easter Day.*	22	
23		23		23		23	
24		24	St. Matthias.	24		24	
25	S. *Sexag.* Conv.	25		25	Annunciation.	25	St. Mark.
26	of S. Paul.	26		26		26	S. *Easter* 5.
27		27		27		27	
28		28		28		28	
29				29	S. *East.* 1, *Low.*	29	
30				30		30	Ascension Day.
31				31			

ALMANAC 9.

Old Style $\binom{\text{XIII., II., X., XVIII.}}{12 \quad 11 \quad 9 \quad 7}$ E 9. New Style (21, 20, 19, 18, 17, 16, 15) E 9.

Days.	January.	Days.	February.	Days.	March.	Days.	April.
1	Circumcision.	1		1		1	
2		2	S. *Sexag.* Purf.	2	S. *Lent* 3.	2	
3		3		3		3	
4		4		4		4	
5	S. *Christm.* 2.	5		5		5	
6	Epiphany.	6		6		6	S. *East.* 1, *Low.*
7		7		7		7	
8		8		8		8	
9		9	S. *Quinquag.*	9	S. *Lent* 4.	9	
10		10		10		10	
11		11		11		11	
12	S. *Epiph.* 1.	12	Ash Wedn.	12		12	
13		13		13		13	S. *Easter* 2.
14		14		14		14	
15		15		15		15	
16		16	S. *Lent* 1.	16	S. *Lent.* 5.	16	
17		17		17		17	
18		18		18		18	
19	S. *Epiph.* 2.	19		19		19	
20		20		20		20	S. *Easter* 3.
21		21		21		21	
22		22		22		22	
23		23	S. *Lent* 2.	23	S. *Lent* 6, *Palm.*	23	
24	[Paul.	24	St. Matthias.	24		24	
25	Conversn. of S.	25		25	Annunciation.	25	St. Mark.
26	S. *Septuag.*	26		26		26	
27		27		27		27	S. *Easter* 4.
28		28		28		28	
29				29		29	
30				30	S. *Easter Day.*	30	
31				31			

167. I now proceed to explain the construction of the *Index* Tables, of which there are *two*—one for Old Style, and one for New Style. First, the Old Style Index Table.

We know (Art. 113) that Dionysius Exiguus took as the Epoch or starting point of the Cycle of the G. Nos. the year B. C. 1, so that A. D. 1 was the second year of the Cycle. Hence, the G. N. of A. D. 1 was II., and the Dionysian Epact 11; we also know (Art. 29) that the Sunday Letter of A. D. 1 was B.

Further, when we know the day of March or April on which Easter falls, it is obvious that we *find* the corresponding *number of the Almanac* by subtracting 21 from the day in March, and adding 10 to the day in April, because Almanac 1 corresponds to March 22; Almanac 2, to March 23; Almanac 11 (= 10 + 1), to April 1, and so on.

Hence, having found by means of Table (Art. 62) the Sunday Letter and G. N. or Nos. corresponding to any of the 35 Easter Days, we find the corresponding Almanac in the first column of that Table. For the Golden Number, the corresponding Dionysian Epact can be at once substituted by means of the series, Art. 165.

TABLE I.

Julian Calendar, or Old Style Index.

Showing for each year of Old Style, from A.D. 1 *onwards, the Dominical Letter, the Golden Number, the Dionysian Epact, and the Number of the Almanac.*

A.D.	Dom. Lett.	Golden No.	Epact.	Almanac.	A.D.	Dom. Lett.	Golden No.	Epact.	Almanac.	A.D.	Dom. Lett.	Golden No.	Epact.	Almanac.
1	B	II.	11	6	29	B	XI.	20	27	57	B	I.	30	20
2	A	III.	22	26	30	A	XII.	1	19	58	A	II.	11	5
3	G	IV.	3	18	31	G	XIII.	12	4	59	G	III.	22	25
4	FE	V.	14	2	32	FE	XIV.	23	23	60	FE	IV.	3	16
5	D	VI.	25	22	33	D	XV.	4	15	61	D	V.	14	8
6	C	VII.	6	14	34	C	XVI.	15	7	62	C	VI.	25	21
7	B	VIII.	17	34	35	B	XVII.	26	20	63	B	VII.	6	13
8	AG	IX.	28	18	36	AG	XVIII.	7	11	64	AG	VIII.	17	32
9	F	X.	9	10	37	F	XIX.	18	31	65	F	IX.	28	24
10	E	XI.	20	30	38	E	I.	30	16	66	E	X.	9	9
11	D	XII.	1	15	39	D	II.	11	8	67	D	XI.	20	29
12	CB	XIII.	12	6	40	CB	III.	22	27	68	CB	XII.	1	20
13	A	XIV.	23	26	41	A	IV.	3	19	69	A	XIII.	12	5
14	G	XV.	4	18	42	G	V.	14	4	70	G	XIV.	23	25
15	F	XVI.	15	3	43	F	VI.	25	24	71	F	XV.	4	17
16	ED	XVII.	26	22	44	ED	VII.	6	15	72	ED	XVI.	15	1
17	C	XVIII.	7	14	45	C	VIII.	17	35	73	C	XVII.	26	21
18	B	XIX.	18	34	46	B	IX.	28	20	74	B	XVIII.	7	13
19	A	I.	30	19	47	A	X.	9	12	75	A	XIX.	18	33
20	GF	II.	11	10	48	GF	XI.	20	31	76	GF	I.	30	17
21	E	III.	22	30	49	E	XII.	1	16	77	E	II.	11	9
22	D	IV.	3	15	50	D	XIII.	12	8	78	D	III.	22	30
23	C	V.	14	7	51	C	XIV.	23	28	79	C	IV.	3	14
24	BA	VI.	25	26	52	BA	XV.	4	12	80	BA	V.	14	5
25	G	VII.	6	11	53	G	XVI.	15	4	81	G	VI.	25	25
26	F	VIII.	17	31	54	F	XVII.	26	24	82	F	VII.	6	10
27	E	IX.	28	23	55	E	XVIII.	7	9	83	E	VIII.	17	30
28	DC	X.	9	7	56	DC	XIX.	18	28	84	DC	IX.	28	21

In the same way the Table may be continued as far as A. D. 532. It need not be continued farther than that year, because the period of 532 years is the Victorian, or Great Paschal Period, of which I have already spoken (Art. 65); at the end of which the Golden Numbers and Dom. Letters occur in exactly the same order and mutual relation as before; each year of the second, third, &c., period having the same Golden Number and Dom. Letter as the 532^d year before it. Hence, the Table for the first 532 years A. D. will apply to any number of subsequent periods. We have only to attend to the following simple Rule:—

"To get the Almanac for any year A. D. which is *beyond* the Table, subtract 532 from it as often as may be necessary in order to come into the Table." For example, suppose we desire to know the Almanac corresponding to A. D. 560, we have $560 - 532 = 28$; the Almanac of which year (28) is 7. Suppose, again, we want to know the Almanac of the year 1582, we have $1582 - 1064 (= 2 \times 532) = 518$, the Almanac of which year the Table shows to be 25; and, in general, if x be any year A. D. higher than 532, the formula is $\left(\frac{x}{532}\right)_r$; the Almanac corresponding to the remainder after this division will be also the Almanac corresponding to A. D. x: *e.g.*, let x be 3085; $\left(\frac{3085}{532}\right)_r = 425$, the Almanac of which year is 29.

If the given year whose Almanac is required be a Leap-year, it must be remembered, as already shown (Art. 165), that the Almanac found by the Table applies only to the ten months from March to December. For January and February of that year we must take the January and February of the *next following* Almanac.

The mode of using the above Table is obvious. Suppose, for example, we require the Almanac for A. D. 66. Inspection of the Table shows that the Almanac is 9; Dom. Letter, E; Golden Number, X.; Epact, 9. Almanac 9 gives Easter Day, March 30, with all the other Moveable and Immoveable Festivals, and any other remarkable days that may be set forth in it. The same Table will serve for any other year which has the same Dom. Letter, and the same Golden Number X., or Epact 9; and moreover, it will serve for any year whose Golden Number is XIII., II., or XVIII.; whose Epact is 12, 11, or 7. Similarly, we find for A. D. 72: Almanac, 1; Dom. Letter, ED; Golden Number, XVI.; Epact, 15. As this was Leap-year, we must look for January and February in the following Almanac, No. 2.

168. We come next to the *Index* Table for the *Gregorian* Calendar, or New Style. This Index is constructed by means of the Gregorian Paschal Table given above (Art. 134). The "Almanac" column shows the number of the Almanac corresponding to

each of the 35 possible dates of Easter, which are given in the fourth column. The "Epact" column shows the different Epacts corresponding to each of the seven Dom. Letters; one of which Epacts must concur with the corresponding Letter in order that Easter Day may fall on one of the 35 days.

Now, we know that *after* the reformation of the Calendar in October, 1582, the Epact of that year was 26 (Art. 123), and the Sunday Letter C (Art. 91). Looking at the Table in Art. 134, we see that Letter C, Epact 26, corresponds to Almanac 28. But *that Almanac will apply only to the remainder of the year after the change took place.* For the next year the Epact was 7, and Dom. Letter B; consequently, we find by the Table, Art. 134, that Easter Day fell on April 10, and that the Almanac was No. 20. Proceeding in this way, the Index Table may be constructed for any number of years after 1582. Francœur has carried it as far as A. D. 2200; and De Morgan as far as A. D. 2000.

The following is a specimen of this Index Table, beginning with 1583 :—

TABLE II.

New Style Index,

Showing, for each Year of New Style from A.D. 1583 *to* A.D. 1676, *the Dominical Letter, Epact, and Number of the Almanac.*

A.D.	Dom. Lett.	Epact.	Almanac.	A.D.	Dom. Lett.	Epact.	Almanac.	A.D.	Dom. Lett.	Epact.	Almanac.	A.D.	Dom. Lett.	Epact.	Almanac.	A.D.	Dom. Lett.	Epact.	Almanac.	A.D.	Dom. Lett.	Epact.	Almanac.
				1597	E	12	16	1613	F	8	17	1629	G	5	25	1645	A	2	26	1661	B	29	27
				98	D	23	1	14	E	19	9	30	F	16	10	46	G	13	11	62	A	10	19
1583	B	7	20	99	C	4	21	15	D	1	29	31	E	27	30	47	F	24	31	63	G	21	4
84	AG	18	11	1600	BA	15	12	16	CB	12	13	32	DC	8	21	48	ED	5	22	64	FE	2	23
85	F	29	31	1	G	26	32	17	A	23	5	33	B	19	6	49	C	16	14	65	D	13	15
86	E	10	16	2	F	7	17	18	G	4	25	34	A	1	26	50	B	27	27	66	C	24	35
87	D	21	8	3	E	18	9	19	F	15	10	35	G	12	18	51	A	8	19	67	B	5	20
88	CB	2	27	4	DC	29	28	20	ED	26	29	36	FE	23	2	52	GF	19	10	68	AG	16	11
89	A	13	12	5	B	10	20	21	C	7	21	37	D	4	22	53	E	1	23	69	F	27	31
90	G	24	32	6	A	21	5	22	B	18	6	38	C	15	14	54	D	12	15	70	E	8	16
91	F	5	24	7	G	2	25	23	A	29	26	39	B	26	34	55	C	23	7	71	D	19	8
92	ED	16	8	8	FE	13	16	24	GF	10	17	40	AG	7	18	56	BA	4	26	72	CB	1	27
93	C	27	28	9	D	24	29	25	E	21	9	41	F	18	10	57	G	15	11	73	A	12	12
94	B	8	20	10	C	5	21	26	D	2	22	42	E	29	30	58	F	26	31	74	G	23	4
95	A	19	5	11	B	16	13	27	C	13	14	43	D	10	15	59	E	7	23	75	F	4	24
96	GF	1	24	12	AG	27	32	28	BA	24	33	44	CB	21	6	60	DC	18	7	76	ED	15	15

The use of this Table is obvious. For example, to find the Almanac for 1598, we look for that year, and find Almanac 1, Dom. Lett. D, Epact 23. The Almanac gives, as before, Easter on March 22nd, the earliest possible day. Again, for 1666, we see the Almanac is 35, Dom. Lett. C, Epact 24. The Almanac gives Easter on April 25th, the latest possible day.

In continuing this Table, we cannot, as in the case of the Old Style Index, make use of the Victorian Paschal Period of 532 years, because it holds only for the unreformed Julian Calendar. The omission of the 10 days in 1582, together with the subsequent Solar and Lunar Equations, interrupts the order of that Period.

Supposing, however, that the above Table has been calculated (as it has been by Francœur) up to A. D. 2200, we may extend it 90 years further by replacing the year 1609 by 2201, 1610 by 2202, and so on; 1699 by 2291; in other words, by continually adding 592. The reason is this, that during those 90 years the Dom. Letters and Epacts are the same in the second series as in the first, nor is it difficult to see why. In the Gregorian Calendar the Dom. Letters reproduce themselves in the same exact order at the end of every 400 years (Art. 95). The remaining 192 years include the centuries 2100 and 2200, in each of which one Leap-year is dropped. Now, in 192 Common years there are 48 Leap-years, which, diminished by 2, become 46. Consequently, the number of regressions is $192 + 46 = 238$, which divided by 7 leaves no remainder. Hence, at the end of those 592 years, the Sunday Letters recur in the same order as before. Again, in those 592 years, the Solar and Lunar Equations reduce the Epacts by 3. But in 592 years there are 31 Cycles of Epacts + 3, as $\left(\frac{592}{19}\right)_r = 3$; and these three being omitted, as just said, the Epacts also will recur in the same order. Accordingly, the Dom. Letters and Epacts of the years from 2201 to 2291 will be the same as those from 1609 to 1699. This does not hold good beyond 1699, because in 1700 a Leap-year is dropped, and the Epact and Dom. Letter of 1700 become 10 and C, respectively; whereas those of 2292 are 9 and CB.

169. The 35 Almanacs are practically useless; for, having found Easter Day (Old or New Style) by any of the previous methods, for any year, we can at once, by means of the Prayer Book "Table of the Moveable Feasts, according to the several days that Easter can possibly fall upon," determine all the Moveable Feasts in that year.

170. To CONSTRUCT THE CALENDAR PRACTICALLY FOR ANY PROPOSED YEAR, OLD OR NEW STYLE.

The two necessary elements are the Dominical Letter and the Golden Number (Old Style), or the Dominical Letter and the Epact (New Style).

The *Dominical Letter* (Old Style) is found either by the Tables (Arts. 29, 30), or by the formula (Art. 32). That for New Style is found by the Table (Art. 95), or by the formula (Art. 92).

The *Golden Number* for both Styles is found by the same formula (Art. 56).

The *Gregorian Epact* is found from the Golden Number, either by the Expanded Table of Epacts (Art. 123) or by the formula (Art. 135).

The Dominical Letter gives the name of the initial day of the year, and consequently the names of all the other days, which are to be written opposite the consecutive days of the month.

The Immoveable Festivals are to be written opposite the fixed days of the month to which they respectively belong.

Easter Day (Old Style) is determined either by the Tables (Arts. 61, 62), or by the formula (Art. 64).

Easter Day (New Style) is found either by the Tables (Arts. 106, &c., and 132) or by the Formula (Art. 137). If the Golden Numbers are used instead of the Epacts (as in our Prayer Book), Easter is found either by the temporary Tables, or by the general Rules in the Prayer Book.

The Moveable Festivals are determined from Easter by means of the Prayer Book definitions.

171. I will conclude by noticing briefly

The Defects of the Gregorian Calendar.

1. The framers of this Calendar made the Tropical year too long by about 27 seconds (Art. 78).

2. The suppression of three Bissextiles every 400 years will require a new correction of a day in about 4000 years. On the other hand, if the more exact length of the year were adopted, and a corresponding suppression of one day every 128 years took place, the uniformity of Bissextile intercalation would be interrupted, and convenience would be sacrificed to greater exactness (Arts. 78, 81).

3. The Lunar Equation of 8 days in 2500 years is not exact. It would be more in accordance with the true length of the synodic month to suppress 5 days in 11 centuries ("L'Art de Verifier les Dates," Dissert. Prelim., § 20).

4. The *Calendar* New Moons do not agree with either the real or mean New Moons: they differ sometimes by as much as three days (Art. 55). But Clavius has shown that a *Cyclical* moon is necessary; and such a moon cannot agree with the real or mean moon, even setting aside the objection that was felt to celebrate Easter on the same day with the Jews.

APPENDIX

ON THE PASCHAL CONTROVERSY.

HAVING explained in Art. 41 the conditions upon which the determination of Easter Day now depends, it may be useful to review in some detail the history of the famous "Easter Controversy" (¹).

From the very earliest times the Christians began to hold *weekly* commemorations of the two great events in the life of Christ—His Crucifixion and His Resurrection. There are several distinct traces in the New Testament writings of a weekly memorial of the Resurrection. And in the second century the observance of the Lord's Day, or Sunday, was universal throughout the Church. Again, at the close of this century, we learn from Tertullian, in the West, and Clemens Alexandrinus in the East, that the so-called *Dies Stationum*, that is to say, Wednesday and Friday, were observed every week; the former, as other ancient writers tell us, in memory of the day of Christ's betrayal to the Sanhedrim; the latter in memory of His Crucifixion (²).

The transition from the weekly to the *yearly* commemoration of the days of Christ's Death and Resurrection was easy and natural. Of all the weeks of the year, that one must have specially aroused feelings of devotion and grateful joy, when the recurrence of the Passover season (³) brought with it the anniversaries of the Death and Resurrection of Jesus. It is not improbable that the

(¹) The first person in modern times who entered minutely into this question was *Gabriel Daniel*, a learned French Jesuit, in his dissertation on the Quarto-decimans (1724). Nearly at the same time, but independently of him, *Chr. Heumann*, a German Professor, wrote a Programme on the same subject. These were followed by Chancellor Mosheim (*de reb. Christ. ante Const.*, pp. 435, *sqq.*), whose results were generally accepted until within a recent date, when the Paschal question was reopened in connexion with the Rationalistic criticism of the Gospels in Germany. The special purpose in view was to deny the genuineness of St. John's Gospel.

(²) Bingham, Book xx., ch. 2; xxi., ch. 3; Guericke, Archäologie, § 23.

(³) "We have every reason to presume that the Rule for the observance of Easter by the primo-primitive Church, at least in Judea and the neighbouring parts of the East, was altogether the same with the Rule observed by the Jews in the celebration of their Passover, before and at the Passion. A desire on the part of the Jews to separate themselves, as widely as possible, from the Nazarenes or Christians, might induce them to alter, *in toto*, the rule for its observance handed down from their forefathers. There is no proof of any such change in the lifetime of St. Paul or St. Peter. But it may have taken place after the destruction of Jerusalem."—Greswell iv., p. 650.

joint celebration of those two intimately associated events took place annually, even during the lifetime of the Apostles themselves (1). St. John and St. Paul, referring to the Jewish Passover, speak of Christ as the true Paschal Lamb (John, xix. 36; Rev. v. 6; 1 Cor. v. 7). St. Paul, in the Epistle to the Romans (iv. 25), combines both events as integral parts of the same great transaction; and in his First Epistle to the Corinthians (xv. 20) he again associates them as antetypes of the Jewish Passover and Wave-sheaf (2). Similarly, the early Fathers, Justin Martyr, Tertullian, Irenæus, Clement of Alexandria, Apollinarius, Hippolytus, speak of Christ's death as the fulfilment of the type of the Jewish Passover (3). So that, although there is no express mention of these anniversaries (or, indeed, of any anniversaries) in the writings of the Apostolic Fathers, it may be concluded, with a high degree of certainty, that from the first they were observed in the Church. The Paschal controversy which arose in the middle of the second century furnishes clear proof that they must have been even then of long standing. It is true, as Mosheim (4) and others have observed, that the early Christian writers generally apply the term *Pascha*, when used by them of the Christian Paschal anniversary, to the Death-day of Christ, without any express intimation that they included a reference to the associated day of the Resurrection (5). But it would be rash to conclude from the silence of the earlier writers on this point that the term Pascha, in its Christian sense, had no such inclusive meaning in their age; and that the anniversary Festival of the Resurrection (which all admit was then celebrated) was at first observed, not as part of the Paschal solemnities, but solely in connexion with the Pentecostal period of fifty days, of which it was the first (6). That there was from very early times a well-recognised connexion between the anniversaries of the Death and Resurrection of Christ seems, as I have

(1) The Jewish Christians, at least, most probably continued from the first the Passover festival, only with a Christian application.

(2) *ἀπαρχὴ τῶν κεκοιμημένων* (1 Cor. xv. 20) is an obvious allusion to *ἀπαρχὴ τοῦ θερισμοῦ* (LXX) in Lev. xxiii. 10. *Clem. Alex.* (in a fragment of his work de Paschate) expressly refers to the Christian analogues of the Passover and Wave-sheaf; *πέπονθε δὲ τῇ ιδ' ὁ Σωτὴρ ἡμῶν, αὐτὸς ὢν τὸ πάσχα* . . . and, in another fragment, *ἐπιμαρτυρεῖ δὲ καὶ ἡ ἀνάστασις· τῇ γοῦν τρίτῃ ἀνέστη ἡμέρᾳ, ἥτις ἦν πρώτη τῶν ἑβδομάδων τοῦ θερισμοῦ, ἐν ᾗ καὶ τὸ δράγμα ἐνομοθετεῖτο προσενεγκεῖν τὸν ἱερέα* (Frag. ii., 1017).

(3) See the passages quoted in Greswell's Dissertations, vol. iii., p. 168–9; Browne, Ord. Sæc., p. 65.

(4) *Vid.* Mosh. *de reb. Christ.*, p. 437, Steitz, *Herzog Encyclop.*, Art. Pascha.

(5) The older writers, ascribing to the word *Pascha* the same meaning as our word *Easter*, which it actually bore in the fourth century, have understood by it the festival of the Resurrection; and hence have represented the question raised about it in the second century to be this: whether the Christians ought to celebrate the *Resurrection* day on the same day as the Jews held their Paschal Feast, *i. e.*, on 14th of Nisan, whatever day of the week this might fall upon; or, inasmuch as Christ undeniably rose on Sunday, the anniversary of this day should be strictly limited to Sunday. This is Bingham's view; *vid.* Book xx., ch. 5, § 2; *vid.* Walshe, i., 667-8.

(6) This is maintained by Steitz in two able articles in the *Stud. und Krit*, 1856 and 1857, and also in the Art. *Pascha*, *Herzog's Encyclopädie*. The true state of the case seems to be that the Resurrection day was not regarded as belonging to either the Paschal or Pentecostal period exclusively, but as having relations with both. Indeed, the latter relation is implied in the reference (above noticed) to the Wave-sheaf, by St. Paul and Clem. Alex. The offering of the sheaf was part of the complete Paschal Festival; and it was also intimately related to Pentecost.

already noticed, to be necessarily implied in the account which we have of the first Paschal controversy. The very closeness of the connexion between these two events, of which the one was the necessary supplement of the others, was, probably, one reason why the early writers did not expressly mention their being component parts of the same annual solemnity. Another reason, no doubt, was the mistake made by the earlier writers referred to of regarding the word πάσχα as etymologically related to *passio*(1); whence would naturally arise their special application of the former word to the anniversary of the Death-day. During the course of the third century the word gradually came to be used in a wider application, and seems to have been indifferently used of the Passion and Resurrection-day. In the early part of the fourth century, and subsequently, we find it applied almost exclusively in connexion with the latter day. The transition period was marked by the distinction made, in fact, if not in express words, between πάσχα σταυρώσιμον and πάσχα ἀναστάσιμον.

No point in Gospel history has been more disputed than the question whether our Lord's Last Supper took place at the same time as the Jewish Paschal Feast, viz., on the evening of the 14th of Nisan, after the Paschal Lamb was sacrificed, or whether it took place on the evening of the 13th; in other words, whether His death took place on the 15th, the day of holy Convocation, or on the 14th, the Jewish Passover day. The difficulty, as is well known, arises from the apparently conflicting statements of the Synoptic Evangelists on the one hand, and St. John on the other. If we had only the first three Gospels, we should certainly conclude that our Lord's Last Supper was identical in time and character with the Jewish Paschal Feast; and, consequently, that His Crucifixion took place on the 15th of Nisan, and His Resurrection on the 17th. Whereas, if we had St. John's Gospel alone, we should as certainly gather from it that our Lord's Supper must have taken place the night before the Passover day, viz., the 13th; that His Crucifixion took place on the day when the Paschal Lamb was slain (14th); that He lay in the grave on the 15th (which was an "high day," *i. e.*, a double Sabbath, because of the coincidence of the weekly Sabbath with the day of holy Convocation); and that the Sunday of the Resurrection was the 16th(2). It is needless to say that numerous attempts have been made to reconcile these discrepancies, and harmonize the Gospel narratives (3). But as yet no perfectly satisfactory solution has been given; and, perhaps, from our want of some necessary data, it may be impossible.

The early Christians were all agreed in keeping their annual commemoration of Christ's Death and Resurrection as nearly as possible at the season of the year at which they actually occurred, that is, at the time of the Jewish Passover. But a certain diversity of practice seems to have prevailed from the very first. There was no dispute among them as to the day of our Lord's Crucifixion having taken place on the 14th of Nisan, the day on which the Paschal Lamb

(1) Augustine, who adopted the true meaning (διάβασις, transitus), rejects the old interpretation *passio*. (*In Joan.*, Tract. iv., Ep. 55, § 1.)

(2) There is no dispute as to the *week*-day of Christ's death. All the Gospel narratives agree in fixing it on Friday, the day before the Jewish weekly Sabbath. The great question is as to the *month*-day, whether the 14th or 15th; and if the 14th, then we must find a year in which the 14th fell on Friday, such as A. D. 29. *Browne*, Ordo Sæcl., pp. 7 and 54.

(3) Some of them may be seen in Smith's *Dictionary of the Bible*, Art. *Passover*. *Browne*, Ordo Sæcl., p. 56.

was slain. The statements of St. John, the tradition of the Primitive Church, the typical connexion between the Paschal Lamb and the Lamb of God, all conspire to leave no doubt in their minds on this head. But starting from this common point, a divergence in practice soon arose, if it did not exist from the beginning. It concerned mainly these two questions:—Firstly, on what day should the commemoration of Christ's Death, and, consequently, that of His Resurrection, be held? Secondly, the proper mode of observing the solemn Fast (1).

I. With respect to the former, the Roman and other Western Churches, and most of the Eastern, laying most stress on the *week*-day, always held their anniversary commemorations on the same days of the week on which Christ died and rose again; namely, on Friday and Sunday; adhering, at the same time, as closely as they could to the time of the Jewish Passover; or, in other words, to the week in which the 14th day of the moon of the first Jewish month took place. It is generally thought that the early Christians, both in the East and West, made use of the Jewish Paschal Cycle of 84 years for the purpose of finding the Crucifixion-day, or the corresponding Resurrection-day. See Bingham, Book xx., ch. v., s. 4. Whenever, for example, the 14th of Nisan fell on a Friday, they agreed with the Asiatics in commemorating the Crucifixion on the 14th, and the Resurrection-day on the 16th (2). But if the 14th fell on any other day of the week, they deferred the commemoration of the Crucifixion to the following Friday (3), and, of course, that of the Resurrection-day to the following Sunday. It may be observed here that the Roman Church from the first seems to have regarded the due celebration of the Resurrection *Sunday* as of prime importance, and to have considered the rest of the Great Week as subordinate to it.

The Christians of Asia Minor, with (according to Athanasius) those of Syria and Mesopotamia, laid, on the other hand, the chief stress on the *month*-day on which Christ died; and, accordingly, always commemorated the day of His Death on the 14th of Nisan, whatever week-day it happened to fall upon, whether Friday or not. They pleaded the authority of St. John and St. Philip in support of their custom; while the Westerns claimed that of St. Peter and St. Paul in defence of theirs.

So far all is clear and unquestioned respecting the first point of difference. But it is not certain, in the first place, whether those Asiatic Christians commemorated the Resurrection on the third day after the Crucifixion (16th of Nisan), whatever day of the week the latter might fall on; *e.g.*, on Friday, if the 14th fall on Wednesday: or whether, like the Westerns, they waited

(1) On this Fast, see Bingham, B. xxi., ch. i. Differences respecting its duration began very early: Irenæus says *οὐ νῦν ἐφ' ἡμῶν γεγονυῖα* (sc. ἡ ποικιλία), *ἀλλὰ καὶ πολὺ πρότερον ἐπὶ τῶν πρὸ ἡμῶν*. Ap. Euseb., lib. v., c. xxiv.

(2) There was no objection felt by the first Christians to keeping their Paschal solemnity on the same day as the Jews kept their Passover. This scruple, which has been the same source of no small difficulty to the framers of the Church Calendar, was of later origin.

(3) The Paschal Chronicle of Hippolytus (A. D. 220) prescribes that if the 14th fall on a Friday, then the Passion-day must be commemorated on that day; otherwise, on the Friday next after the 14th. This would, obviously, sometimes occasion the difference of an entire week in the commemoration of the Crucifixion, and also of the Resurrection. When the 14th fell on Saturday, the Asiatics kept their commemoration of Christ's death on that day, whereas the Westerns deferred theirs to the following Friday.

for the Sunday next after the 14th. The former is the generally received opinion (1). The determination of this question mainly depends on the interpretation of the well-known passage in Eusebius (E. H., v. 23), which, though embarrassed with some difficulties, seems very strongly in favour of the first view (2).

II. We come now to the second point at issue between the Christians of Asia Minor and the Westerns, namely, that connected with the Fast of the Great Week.

The ground of this difference was the different view taken of the tone of feeling suitable to the day of Christ's death. The Westerns regarded the day as exclusively one of gloom and mourning. They regarded it solely under its *historical* aspect, putting themselves into the same attitude of profound sorrow in which His bereaved followers were on the day of the Crucifixion. The Asiatics, on the other hand, looked at the day from its *dogmatic* side also, as the day of the redemption of the human race; and, therefore, it was to them not wholly a day of mourning, but also of rejoicing. The hours of that day which preceded His death were spent in mourning and fasting; but from the moment when the Divine Victim expired, and the atoning sacrifice was complete, all was changed—sorrow was turned into joy, mourning gave place to exultation. Accordingly, they ended their Fast at the hour when Christ gave up the ghost, 3 p.m. (the ninth hour); and then commenced their Paschal Feast (3). The Westerns, on the contrary, regarding the Crucifixion day as wholly one of mourning, continued their Fast throughout the entire of that day and the following day, in which Christ lay in the grave, and did not end it until the morning of the Resurrection, when they celebrated their Paschal Feast. In each case the Feast included the δεῖπνον κυριακὸν, and, in very early times, the accompanying ἀγαπὴ (4).

(1) There is yet a third view, differing from both the above; namely, that the Asiatics did not observe an *anniversary* festival of the Resurrection at all, contenting themselves with the *weekly* commemoration, every Sunday. Giescler maintains this view, *Kirchengeschichte*, i., p. 241. He is followed by Steitz, *Stud. und Krit.*, l. c.

(2) The passage referred to is this:—*Σύνοδοι δὴ καὶ συγκροτήσεις ἐπισκόπων ἐπὶ ταὐτὸν ἐγίνοντο, πάντες τε μιᾷ γνώμῃ δι' ἐπιστολῶν ἐκκλησιαστικὸν δόγμα τοῖς πανταχόσε διετυποῦντο, ὡς ἂν μηδ' ἐν ἄλλῃ ποτὲ τῆς κυριακῆς ἡμέρᾳ τὸ τῆς ἐκ νεκρῶν ἀναστάσεως ἐπιτελοῖτο τοῦ Κυρίου μυστήριον, καὶ ὅπως ἐν ταύτῃ μόνῃ τῶν κατὰ τὸ πάσχα νηστειῶν φυλασσώμεθα τὰς ἐπιλύσεις.* If by the words *τὸ τῆς . . . μυστήριον* is to be understood the Resurrection itself, then it follows that this event was by some commemorated on a day different from Sunday. To escape this conclusion, *Weitzel* (die Christl. Passafeier) takes *μυστήριον* to mean *Sacrament*, and maintains that what the Bishops meant was, that the Resurrection *Communion* should be celebrated on no other day than Sunday; whilst the Asiatics commemorated the event (as the Westerns did) on Sunday, but held the Communion on the 16th.

(3) In the statement just made I have followed the view now usually held. But it must be confessed that it rests mainly on what I can hardly help believing to be a false interpretation of the word *κατὰ* near the beginning of the passage of Eusebius, lib. v., ch. 23. As generally interpreted, the words *κατὰ ταύτην* are explained as meaning that the Asiatic Church broke off the Fast *on* the 14th day of Nisan. But the true meaning seems to be this, that, as the Asiatics kept the 14th day, on whatever day of the week it might fall, so they also broke off the Fast at the time *corresponding to* (*κατὰ*) the 14th; that is to say, probably, on the morning of the 16th, when they celebrated the Festival of the Resurrection. See *Mayer*, Echtheit des Evang., n. Johan., *s.* 394.

(4) Mosheim (*de reb. Christ. ante Const.*, p. 443) says that the Westerns celebrated their Paschal Feast on the night of Saturday, and not on the morning of Sunday. The

Another disputed point, but one of secondary importance, respecting the Fast, is this,—whether the Asiatics entirely ended their Fast at 3 p.m. on the 14th, or merely interrupted it, resuming it again as soon as their Paschal Feast was concluded, and continuing it until the Resurrection Feast began. The words of Eusebius, v. 23—τὰς νηστείας ἐπιλύεσθαι, τὰς τῶν νηστειῶν ἐπιλύσεις—seem clearly to imply a total end, and to be inapplicable to a mere interruption. Mosheim indeed (*l. c.*, p. 441) endeavours to prove from a passage of Epiphanius (Hær. lxx., § 11) that the Andiari, a sect of Asiatic Quarto-decimans, again fasted after their Paschal Feast. But even admitting that they did so, it would not be legitimate to infer from it the general practice of the Asiatics. But the truth is, that the passage of Epiphanius, when properly interpreted, does not yield the sense which Mosheim tries to elicit from it.

From their custom of commemorating the Death-day of Christ on the 14th of Nisan, the Minor Asiatic Christians were designated *Quartodecimans.* Until a very recent period it was usual with Church historians to consider that all Quartodecimans belonged to the same party, and held the same religious tenets. But recent investigations seem to have clearly proved that there were at least two very distinct bodies to whom the ancients applied this term—one, the Asiatic Christians of whom I have been speaking, who belonged to the Catholic Church, and stood on Christian, not on Jewish ground: the other party belonged to the Judaizing sect of the *Ebionites.* Both parties agreed in keeping the same day as the Jews, 14th Nisan; but they differed essentially both as to the reason for keeping that day, and the mode of observing it. The Ebionite party maintained the abiding obligation of the Old Testament ritual for Christians, and, accordingly, the necessity of eating the Paschal Lamb, like the Jews, on the 14th. The orthodox Quartodecimans, on the other hand, held that the old Paschal Festival was abolished, the death of Christ on the Cross having done away with the type that prefigured it; and, therefore, they commemorated, not the Passover, but the death of Christ. Both parties appealed to the Gospels. The Ebionites asserted that Christ Himself had on the 14th of Nisan eaten the Paschal Lamb, and was not crucified until the 15th; and that all Christians were bound to follow His example. The orthodox Quartodecimans, on the contrary, held that Christ did not eat the Paschal Lamb the year He suffered, but was crucified on the 14th, before the hour when the Jewish Paschal Feast began; and, accordingly, that the 14th was the anniversary, not of the Paschal Feast, but of His death (1).

The first recorded occasion on which the difference between the orthodox parties respecting the Paschal question came to a formal discussion was about the middle of the second century (A. D. 158), when Polycarp, Bishop of Smyrna, visited Rome, in the Popedom of Anicetus. They separated amicably, each party adhering to its own custom, and continuing to hold communion with the other. About forty years later (A. D. 198), the controversy was renewed, in a much

passage which he quotes in proof from Epiphanius does not sustain his inference. And, besides, all the evidence is the other way. See Bingham, B. xxi., ch. 1, § 32.

(1) In the above account of the nature of the old Paschal Controversy I have chiefly followed Hefele, *Conciliengeschichte*, vol. i., p. 286, *sq.*

more bitter spirit by Victor, Bishop of Rome, against Polycrates, Bishop of Ephesus. Several Councils were held in the East and West, and decided in favour of Rome. The Asiatics, however, refused to give up their traditional usage. Polycrates wrote to Victor, pleading the authority of the two Apostles, St. John and St. Philip, and of other high names, but in vain. Victor excommunicated the Asiatics, and tried to procure a like condemnation of them from other Churches. In this, however, he did not succeed. Through the interference and mediation of several Bishops, especially Irenæus (1) peace was at last restored, and the Asiatics continued to observe their ancient usage until the time of the Nicene Council. The differences just described, which properly constituted the subject-matter of the celebrated "Paschal Controversy," turned, as we have said, mainly on two points: First, as to whether, in the commemoration of the Paschal solemnities the week-day (Friday), or the month-day (14th Nisan) should be preferred, in case they did not coincide; and, in connexion with this, whether the Resurrection-day ought to be commemorated on any day but Sunday; and, secondly, whether it was lawful to break off, or interrupt, the Fast of the Great Week before the Resurrection-day.

In the course of the third century a new element of disagreement was introduced by the different Paschal canons which then appeared. It is generally agreed that the first Christians, both in the East and West, made use of the old Jewish Lunar Cycle of eighty-four years in order to determine the time of the Paschal anniversaries (2), according to which Cycle the Passover was regulated by the first Full Moon (*i.e.*, the fourteenth day after the appearance (3) of the New

(1) Irenæus was a native of Asia Minor, and in his youth had personally known Polycarp. Connected, then, as he was with the East and West, in his early life a Quartodeciman, and in his own Church of Lyons a follower of the Roman usage, he was specially fitted to play the part of a mediator and peacemaker.

(2) *Ideler* (i., 570; ii., 243) strongly expresses his opinion that up to the time of the Captivity, and even long after, the Jews determined the New Moon (including of course the Paschal) by direct *observation*, without any fixed astronomical rules. *Riccioli* (Chron. Reform. B. i., c. 12) gives strong reasons for believing that this continued down to the time immediately preceding the Christian era. Epiphanius, however (Hœr. 51, c. 26), speaks of an eighty-four-year Cycle, by means of which the Jews, at the time of Christ, determined the Passover day. It is unmistakable that there is no trace of such a Cycle in the Talmud, or in any Rabbinical writer. However, Cyril of Alexandria also mentions (in his Prologus Paschalis) that the Latin Church made use of an eighty-four-year Cycle before their adoption of the Paschal canon of Hippolytus. This Cycle was probably made use of by some of the Jewish sects, and from them adopted by the Quartodeciman Christians of the East. How and when it came into use in the West we have no information. But it seems clear that the early Roman Church regulated their Easter by it for a considerable time; until it was temporarily displaced by the Easter canon of Hippolytus, in the third century. How long Hippolytus' canon continued in use at Rome we do not know. But there is no doubt that at the beginning of the fourth century an eighty-four-year Cycle was again employed there (*Ideler*, ii., 238).

(3) The Mosaic definition of the time of celebrating the Passover (as given in Ex. xii. 2, 6; Lev. xxiii. 5; Num. xxviii. 16) is this:—"the 14th day of the first month, at evening (or between the two evenings"). We need not enter on the much debated question as to whether the Jewish year at that time was Solar or Lunar. Usher contends for the former, and Petavius is inclined to follow him. Most modern authorities maintain the latter. However this may be, the "*First month*" (called "*Abib*" in Ex. xiii. 4, but changed to "*Nisan*"

Moon) which fell after the Vernal Equinox (1). The Christians continued to employ this Jewish Cycle for about a century; when, partly because of some supposed errors in it, but, no doubt, chiefly in order to render themselves independent, in a matter of such importance, of the aid of those whom they despised and hated, they set about framing Lunar Cycles for themselves. The earliest of such Cycles now extant is that drawn by Hippolytus, Bishop of Portus, at the mouth of the Tiber (2). Its date is A. D. 220; and it is a most important record of the practice of the Western Church, especially that of Rome, in the early part of the third century. It was a sixteen years' Cycle (ἑκκαιδεκαετηρὶς); in other words, he supposed that the New Moons fell on the same days of the month at the end of every sixteen years, and on the same days of the week and month at the end of every 112 years; and he further assumed that the Equinox took place on March 18. The oldest extant Greek Paschal canon is that of Dionysius of Alexandria, published in a Festal Epistle (*circ.* 250). It is lost, and we know nothing more about it than what Eusebius tells us (lib. vii., c. 20), viz., that it was calculated by an eight-year Cycle (ὀκταετηρὶς) (3), and that it particularly specified that the Paschal Feast should not be kept until after the Vernal Equinox, which with him was March 21. Soon after (*circ.* 277), Anatolius, Bishop of Laodicea, an Alexandrian, whom Eusebius (vii. 32) describes as a great geometer and astronomer, and otherwise the most learned man of the age, made use of the famous Metonic Cycle of nineteen years (ἐννεακαιδεκαετηρὶς) in constructing his Paschal canon, taking for the Equinox the 19th of March. He proved from several ancient Jewish writers themselves that the Passover

by the later Jews, who borrowed the name from the Assyrians and Babylonians), was that month the fourteenth day of whose Moon fell either *on* the day of the Vernal Equinox, or on the nearest day after it (Clavius, p. 58). It is important to observe that the fourteenth day of the moon was reckoned, not from conjunction, but from the first *appearance* (phasis) of the New Moon, which in fine climates like that of Palestine would usually take place about eighteen hours after conjunction, and just after sunset; and as the Synodic month is about 29^{d} 12^{h} long, the fourteenth day reckoned in this way would very nearly coincide with the actual Full Moon. Persons were appointed by the Sanhedrim to watch the first appearance of the New Moon, each month; and when it was seen, public proclamation was made by the proper authorities that the month had commenced. A similar custom prevailed, as we have already seen, among the ancient Romans, according to Macrobius (i. 15). The ancient Greeks, also, whose year was Lunar, reckoned in the same way. The first day of the month (νουμηνία) was not the day of conjunction, but the day on the evening of which the New Moon was first visible.

(1) Bingham, B. xx., ch. v., § 4. Ideler, *Chronol.*, ii., 228. The rule observed by the Jews of the fourth century seems to have been directly the reverse of that followed by the earlier Jews: the latter invariably celebrated their Passover *after* the Vernal Equinox (*vid.* Greswell, *Diss.*, vol. i., pp. 316–319, 327–329): the former before it (*ib.*, vol. iv., pp. 643 and 650).

(2) *Vid.* Bunsen, Hippolytus i., p. 240. This canon of Hippolytus (mentioned by Eusebius, *E. H.*, lib. vi., ch. 22) was no doubt based on the eight-year Cycle used long before by the Greek astronomers, a description of which is given by Geminus (*vid. Ideler*, i. 294). Respecting this canon of Hippolytus, full information will be found in *Ideler*, ii., 213, *sqq.*, and in the older works to which he refers.

Recently, an ingenious argument has been drawn from this canon by Dr. Salmon, with the view of proving that Hippolytus was the author of the Liberian Catalogue, which occupies an important place in the question of the succession of the first Bishops of Rome. *Vid.* Hermathena, No. i., p. 82, *sqq.*

(3) On the Octaëteris of Dionysius, *vid.* Browne, p. 481.

should never be kept before the Vernal Equinox, and therefore that their Cycle was erroneous (1). Anatolius' Cycle and canon were adopted at Alexandria, with this important alteration, that the Equinox was assigned to the 21st of March. From all this it happened that, at the beginning of the fourth century, the uncertainty and difference respecting the time of celebrating Easter (2) were even greater than before. Some disregarded the Equinox altogether, and celebrated Easter before or after it indifferently. And to say nothing of other minor cases, we find the two greatest Churches of Christendom, Rome and Alexandria, calculating Easter on different methods. The Alexandrian Cycle of nineteen years, with the Equinox on March 21, must necessarily often give a result differing from the Roman (Hippolytan) Cycle of sixteen years, with the Equinox on the 18th of March. For example, when in any year the fourteenth of the moon fell on the 19th of March, the Romans (and those who reckoned with them) considered it to be the Paschal moon, and held their Easter Festival accordingly, whilst, according to the Alexandrian reckoning, the Equinox had not yet arrived; and therefore they did not keep their Easter Festival until the fourteenth of the following moon, that is to say, a month later than the Westerns.

These many and great differences respecting the Easter Festival gave rise, especially in countries adjacent to each other, to much perplexity and many collisions, and exposed the Christians to the scorn and derision of the heathen (Epiphan., Hær., lxx., 14). In the West, the first Council of Arles (314) attempted to bring about uniformity in this matter, and to assimilate the different usages to the Roman, by decreeing in its first canon, "ut Pascha Dominicum uno die et uno tempore per omnem orbem a nobis observetur" (Masi, tom. ii., p. 471). This decree failed to produce the desired effect. And, accordingly, the Council of Nice (325), at the instance of the Emperor Constantine, took the matter in hand. We have no detailed account of the deliberations of the Council on this subject, nor is there any allusion made to it in the existing twenty canons of the Council (3). Our only information respecting it is derived partly from the Encyclical Epistle addressed by the Council to the Churches of Alexandria, Libya, and the Pentapolis (4), partly from a circular letter written by Constantine himself to the Bishops, who did not

(1) The passage in the *Constitutiones Apostolicæ* is strikingly coincident with Anatolius' canon, and makes it highly probable that it was written after the canon Greswell, Dissert. iv., p. 639.

(2) "Pascha" at this time undoubtedly denoted the Resurrection Festival, and it seems equally certain that the fourteenth day of the moon was now, by the great majority of Christians, in the East as well as in the West, considered chiefly, if not solely, in connexion with the Resurrection Day; and not, as originally, in connexion with the Passion Day. When this took place, or by what steps it was effected, does not appear. It is certain that, subsequent to the Council of Nice, Easter Day was regulated by this fourteenth day: and so completely was all reference to the Passion Day given up, that if the Equinox and also the fourteenth of the moon fell on a Saturday, the following day was kept as Easter Day.

(3) Almost all the acts of the Nicene Council, except those relating to the Arian controversy, are involved in much obscurity. Even the very date of the meeting of the Council has been differently given by the early historians. The number of the canons which are extant in the original Greek and in the contemporary Latin versions is twenty. Vid. *Routh*, Script. Eccles. Opusc., pp. 354–367: and cf. Greswell, Dissert., iv., 674.

(4) *Socrates*, Hist. Eccles., lib. i., c. 9. *Theodorit*, Hist. Eccl., lib. i., c. 9.

attend the Council (1). The Council says, in the Epistle referred to, "We also send you good news of the unanimity respecting the most holy Pascha, that this matter has been settled by means of your prayers. So that all the brethren in the East, who hitherto celebrated it at the same time with the Jews, will in future conform to the Romans and to us, and to all who have kept the Pascha with us."(2) It is important to remark that the Council does not, either here or elsewhere, lay down any explicit rule for determining Easter. It merely says, in general terms, that henceforth the Eastern Churches should conform to the practice of the Romans and of the other Churches which agreed with them. They obviously meant to confirm the rule then most generally prevalent, namely, that the Resurrection Day should (in opposition to the Quarto-decimans) always be kept on *Sunday;* that the fourteenth day of the Paschal moon should *follow* the Equinox; and that Easter Sunday should *follow* the fourteenth day of the moon (3). That this latter point was especially intended to be insisted on by the Council seems to be proved from the Letter of Constantine above referred to, in which he states that the Council, besides unanimously decreeing that Easter should everywhere be kept on the *same* day, also resolved that in celebrating the most holy Feast, it was a thing wholly unworthy of the Christians to follow the usage of the Jews:—μηδὲν ἔστω ἡμῖν κοινὸν μετὰ τοῦ ἐχθίστου τῶν Ἰουδαίων ὄχλου. Accordingly, if the fourteenth of the Paschal moon happened to fall on Sunday, Easter Day was not kept till the Sunday after, in order to avoid a coincidence with the Jewish Passover (4).

It has been generally maintained by Roman Catholic, and acquiesced in by many Protestant, writers, that the Council of Nicæa did lay down some express rules for the determination of Easter; in fact, that it propounded the actual rules which were in use in the Church for that purpose down to the Gregorian reformation of the Calendar in 1582 (5). The authority chiefly

(1) *Eusebius*, Vit. Constant., iii., 18; *Theodorit.*, i., 10.

(2) Constantine, in his letter, enumerates Italy, Africa, Ægypt, Spain, Gaul, Britain, Libya, Greece, Asia, Pontus, and Cilicia.

(3) In a decretal Epistle attributed to Pope Pius (*circ.* A. D. 159) it is laid down, in opposition to the Quarto-decimans, that Easter should be kept only on *Sunday*. And Pope Victor (A. D. 198), in his Epistle to Theophilus, Bishop of Cæsarea, further declares that Easter Sunday must follow the fourteenth day of the first month; *i. e.*, the month the fourteenth day of whose moon falls between the 21st of March and the 18th of April, both inclusive. And the same was confirmed in the East by the authority of the Council of Cæsarea. Vid. *Clavius*, cap. i., 15.

(4) There does not appear to have been any such scruple in earlier times respecting an agreement with the Jews in this matter. At all events, it did not extend to the day of our Lord's Crucifixion. For the Paschal Chronicle of Hippolytus Portuensis prescribes, that if the fourteenth day of the moon fall on a Friday, then the Passover must be celebrated that day (the day of the Jewish Passover); otherwise, on the Friday next after the fourteenth.

Gresswell (*Dissert.*, iv. 672) suggests what seems a more probable reason for the rule, that if the fourteenth of the moon fell on Sunday, Easter should not be celebrated till the following Sunday; namely, that in such a case the commemoration of the Passion Day must be deferred till the following Friday, because the Fast could not be kept on Sunday, it being peculiar to the Manichæan heretics to fast on Sunday.

(5) Roman Catholics naturally desire to claim for the Easter rules, so long observed in the Church, the authority of the most famous of all the ancient Councils. Protestants, also, regarding the Nicene Council as antecedent to the assertion of papal supremacy, are

relied on in support of this opinion is that of St. Ambrose, Archbishop of Milan, who, in a letter written about sixty years after the Council (1), states that the Nicene Council, assisted by a number of persons skilled in astronomical calculations, decreed that Easter should be celebrated after the fourteenth moon of the first month; and that this fourteenth moon should be determined by the Lunar Cycle of nineteen years. In support of this statement the testimonies of Cyril and Proterius, Bishops of Alexandria in the middle of the fifth century, are appealed to (2), and especially that of Dionysius Exiguus, the famous author of the Vulgar Era, in two Epistles written by him, A. D. 525 and 526 (3). It seems hard to disbelieve such positive testimony as to

glad to refer to that Council the establishment of the rules which they themselves continued to observe. The better informed Roman Catholic writers content themselves with saying that the Nicene Council only *confirmed* the practice already prevailing in the Roman and most other Churches. Thus, *Clavius* (cap. i., p. 60) writes:—" Etsi autem verba illa Nicæni Concilii expresse non explicant quo die Pascha celebrandum sit, sed solum cum Judæis non esse observandum indicare videantur, non obscurè tamen ex his intelligi potest consuetudinem Romanæ Ecclesiæ, sanctionemque Pii ac Victoris . . . confirmatam esse atque renovatam." In another passage (cap. v., § 10) Clavius expressly states that the reason why so many disputes continued to arise between the Greeks and Latins concerning the proper time of keeping Easter, even after the Nicene Council, was this, that the Nicene Fathers had not enjoined the use of any particular Cycle, which the whole Church should follow, but committed the task of determining this point to Eusebius and the Alexandrians. It must, however, be admitted that Clavius does not seem quite consistent with himself in this matter, because in a later chapter (xi., § 5) he asserts that "the Cycle of the Golden Numbers was arranged by the Fathers of the Nicene Council." He probably meant only that it was afterwards done, in accordance with their *general* directions.

(1) *Ambros.* Oper. ii., 880, Epist. 23, ad Episc. per Æmil. const.: " Non mediocris esse sapientiæ diem celebritatis definire Paschalis, et Scriptura divina nos instituit, et eruditio majorum, qui convenientes ad Synodum Nicænam . . . congregatis peritissimis calculandi, decem et novem annorum colligere rationem, et quasi quendam constituere circulum, ex quo exemplum in annos reliquos gigneretur. Hunc circulum ἐννεαδεκαετηρίδα nuncuparunt. . . Duo autem sunt observanda in solennitate Paschæ, quartadecima Luna et primus mensis. . . incipit autem mensis, non secundum vulgarem usum sed secundum consuetudinemperitorum, ab Æquinoctio, qui dies est xii. Kal. April. (March 21); et finitur xi. Kal. Maii" (April 21). Professor de Morgan (Companion to the Almanac, 1845, p. 7) mistakes in supposing that Clavius represents Ambrose as stating that the Cycle was arranged for the purpose by Eusebius of Cæsarea. The statement about Eusebius is not Ambrose's at all, but merely Clavius' own inference from Ambrose's words "congregatis peritissimis calculandi," which Clavius thought were intended to apply to Eusebius and the Alexandrian astronomers. This inference Clavius goes on to support by the express testimony of Bede (de Tempor. Rat. c. 42): " Decennovennalis circuli ordinem primus Eusebius ad quartasdecimas Lunas Festi Paschalis, ipsumque diem Paschæ inveniendum, composuit." Bede seems to have followed Jerome, who erroneously asserts (de Scriptor. Eccles., lxi.; Oper., iv., pars ii.) that Eusebius was the author of the nineteen-year Cycle.

(2) Cyril's words, in an Epistle written to the Council of Carthage, A. D. 433, are: "Scruteminique diligentissime, quæ ordinavit Synodus Nicæna, Lunas xiv. omnium annorum per Decemnovennalem Cyclum."

Proterius (A. D. 453) writes: "Beatissimi Patres nostri, Cyclum Decemnovennem certius affigentes quam violari impossibile est, velut crepidinem ac fundamentum et regulam hunc eundem Decemnovennalem computum statuerunt."

(3) Dionysius Exiguus, in his two Epistles, written A. D. 525 and 526, as preliminary or supplemental to his Paschal Cycle, says " Paschalis Festi rationem explicare curavimus, sequentes per omnia venerabilium trecentorum et octodecim Pontificum qui apud Nicæam convenerunt, etiam rei hujus absolutam veramque sententiam: qui quartasdecimas Lunas Paschalis obser-

the fact of the Council having propounded some precise rules on the subjects of Easter. But, on the other hand, we are met with the difficulties arising from the total silence of the extant records of the Council respecting any such rules, joined to the equally perplexing fact that for many years after the date of the Council, differences continued to exist, and violent dissensions disturbed the peace of the Church, on the very subject of the observance of Easter, which the great Œcumenical Synod is supposed to have finally settled (1).

The usual way in which these conflicting difficulties are attempted to be reconciled is, by supposing that the Council did not itself put forth any explicit rule for the *computation* of Easter, but only commissioned (2) the Church of Alexandria, as most eminent in astronomical science, to make the calculations and frame a rule based on the Cycle of nineteen years; so that the Council might be said to have, by anticipation, sanctioned the Paschal Rule subsequently drawn up by that Church. At all events, the Paschal controversy was not settled by the Council of Nicæa. A few years after, the Council of Antioch (A.D. 341) found it necessary to take up the matter, and excommunicated the adherents of the Jewish usage. Again, the Council of Laodicæa (A.D. 364), and the second General Council (Constantinople, A.D. 381) condemned the Quartodeciman heresy, which did not finally die out until the sixth century. But even the Latin and Alexandrian Churches still occasionally differed. The very year after the Council of Nicæa, and also in 330, 333, 340, 341, 343, the Latins celebrated Easter on a different day from the Alexandrians (3). In the year 387, the former kept Easter on the 21st of March, while the latter kept it on the 25th

vantiæ per novemdecem annorum redeuntem semper in se circulum stabiles immotasque fixerunt . . . deinceps venerabilis Theophilus, et Cyrellus, ab hac Synodi veneranda constitutione minime desciverunt; imo potius eundem decemnovennalem circulum solicite retinentes." Dionysius, moreover, appeals to the source from which he derived his knowledge of these facts; namely, the "Seventh book of Ecclesiastical History," a work not now known to exist, but which seems to have contained some account of the Paschal Cycle of the Council. Vid. *Greswell*, Dissert. iv., pp. 698–707; and *Bucherius*, de Doctr. Tempor.

(1) In answer to the latter difficulty, Greswell says (iv., 697): "When we consider the confusion prevalent in the Church at the time of the Council of Nice, and from thenceforward to the accession of Theodosius, A.D. 379, when tranquillity was first restored; when we take into account the opposition that was made to its decrees on other, and much more important, points, it will not be surprising that a Paschal Cycle, though published with the sanction of the Council, should not have obtained an universal or uniform reception all over the Church."

Greswell argues (vol. iv., pp. 662, *sq.*) at great length, and with some force, that the Nicene Council did actually draw up the rules respecting the observance of Easter.

(2) The earliest mention of the task being deputed to the Church of Alexandria to calculate Easter for the observance of the rest of Christendom seems to be in the Prologue of Cyril of Alexandria to his Paschal Cycle of ninety-five years (A.D. 437). The next distinct allusion to it is the Epistle of Pope Leo the Great to the Emperor Marcian (A.D. 453). Neither Cyril nor Leo mentions by name the General Council which deputed this authority. Greswell (iv., 390) thinks it may have been, not the Council of Nice, but the Second General Council, that of Constantinople (A.D. 381).

(3) Vid. *Hefele*, Conciliengesch., i. 315; *Ideler*, ii. 253. The cause of this appears to have been the fact, established by Cardinal *Noris* and *Muratori*, that while the Alexandrians employed the nineteen-year Cycle, the Romans adhered to the Cycle of eighty-four years, with the Easter limits of the 19th of March and 21st of April. The consequence of this different mode of computation was, that during the course of a single Cycle (298–381) the Latins differed from the Alexandrians no less than

of April. This divergence of five weeks (the greatest possible) induced the Emperor Theodosius to request Theophilus, Bishop of Alexandria, to prepare a common Easter Table for the use of both Churches, which he accordingly did (1). Nevertheless, about fifty years after, a difference of a month again occurred between the two Churches, which gave occasion to Cyril, the then Patriarch of Alexandria, to construct a new Paschal Table of ninety-five years, being an abridgment epitome of the larger Table (which is lost) of his uncle and predecessor, Theophilus. But not even did this put an end to mistakes and misunderstandings upon the subject. A most important step towards finally removing them was made by Victorius (2), Bishop of Aquitaine (*circ.* A. D. 457), who, at the request of Hilary the Roman Deacon, composed a new Cycle, combining the Alexandrian Lunar Cycle of nineteen years with the Solar Cycle of twenty-eight years. This was the famous *Victorian Period* of 532 years, which subsequently received the name of the Dionysian Period, because Dionysius Exiguus adjusted it to his new era of the Nativity. Hilary seems to have adopted the Victorian Period when he became Pope (A. D. 465). Still, discrepancies occurred. Six times within the course of forty-one years (475–516) Easter was celebrated eight days later in the West than in the East. At length, the Roman Abbot Dionysius Exiguus, a native of Scythia, had the merit and glory of finally settling this long disputed question. It is said that he did so by command of the Emperor Justinian. He employed the Victorian Period of 532 years, in the construction of his own Easter Table of ninety-five years, commencing with A. D. 532 (532 to 627). In fact, as has been just observed, he seems to have done little more than adapt the Victorian Period to his new era of the Nativity, which he substituted for the Alexandrian era of Diocletian, hitherto used by Christians. This ninety-five years Paschal Table of Dionysius met with almost universal reception in the East and West. Isidore of Seville continued it for ninety-five years longer (627–721); the Venerable Bede carried it on to A. D. 1063, and thus completed the whole Period of 532 years (532–1063). It did not, however, entirely displace the eighty-four year Cycle and the Victorian Table until about the time of Charlemagne. Spain did not wholly conform to the Dionysian reckoning till the year 587; Britain not till A. D. 729 (3); Gaul not till the close of the eighth century. Thus terminated the controversies about Easter, which for more than four centuries agitated the Church both in the East and West, and were not entirely extinguished for upwards of two centuries more.

Thenceforward the subject remained in a state of complete quiescence until the thirteenth

twenty-two times in the observance of Easter. Vid. *Herzog*, Encyclop., vol. xviii., 480.

(1) Ideler, ii., 254.

(2) Not Victor, or Victorinus, as some modern chronologists write the name.

(3) The Saxons, on their conversion by Augustine, at the close of the sixth century, adopted the Dionysian reckoning. But the Britains adhered, for more than a century longer, to the older usage of the Roman Church, to which, it is recorded, they conformed in the Popedom of Leo the Great. The early British Churches, like the Asiatic Quartodecimans, professed to derive their practice from the Apostle John. Yet, they were not Quartodecimans, inasmuch as they kept Easter always on a Sunday. British Bishops sat, we know, in the Council of Arles (A. D. 341), and most probably concurred in the Decree of that Council, approving the Roman rule respecting Easter. Vid. *Robertson*, Hist. of the Church, i., 588.

century, when what has been termed the "Achilles heel" of the Old Church Calendar (1) was first exposed by an unknown writer in the year 1223 (2). Thus revived, the subject continued to be from time to time discussed, until it was again, and finally, set at rest by the Gregorian reformation of the Calendar, in the year 1582.

(1) The vulnerable point in the Church Calendar, as arranged by Dionysius, was the two false assumptions on which it was constructed; namely, that the Julian year is exactly equal to the mean Tropical year, and that the Lunar Cycle of 235 Lunations in nineteen Julian years is also perfectly exact.

(2) The author was generally supposed to be Vincentius, of Beauvais, in France.

THE END.